On the Variants of Subset Sum: Projected and Unbounded

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- Randomized $\tilde{O}(n + t)$ algorithm for Unique Projection Subset Sum

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- Reduction from Unbounded Subset Sum to Closest Vector Problem

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$$f(x) = 1 + x^{a_1} + \ldots + x^{a_n} + x^{a_1+a_2} + x^{a_1+a_3} + \ldots + x^{a_1+a_2+\ldots+a_n}$$

Randomized $\tilde{O}(n + t)$ algorithm for Unique Projection Subset Sum

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Claim: $t' \leq 2t$

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To find j, use a different polynomial and binary search.

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$$g(x) = \prod_{i=1}^{n/2} (1 + 2x^{a_i}) \prod_{i=n/2+1}^{n} (1 + x^{a_i})$$

Reduction from Unbounded Subset Sum to Closest Vector Problem

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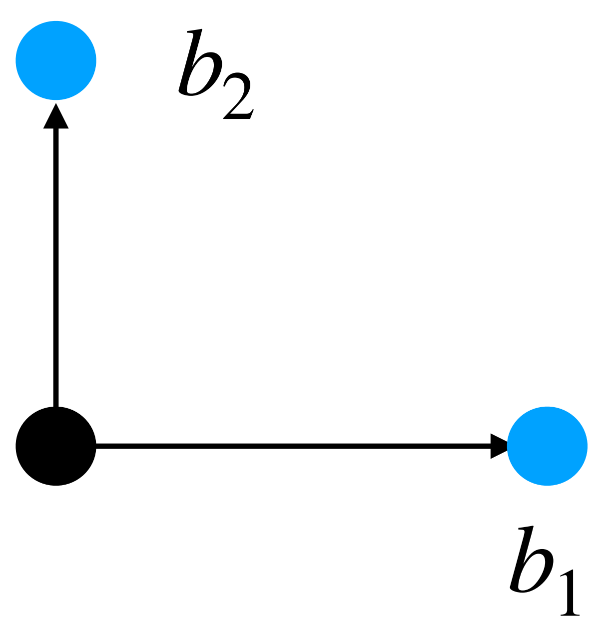
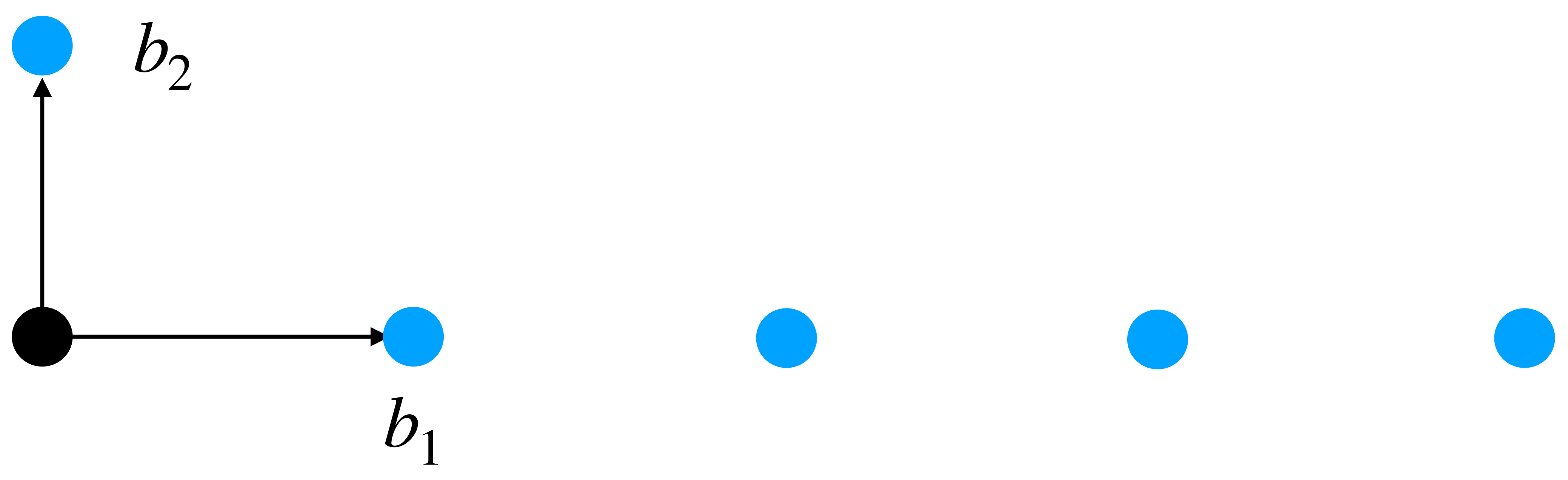
B is called a *basis* of $\mathscr{L}$.


b_2
b_1


b_2
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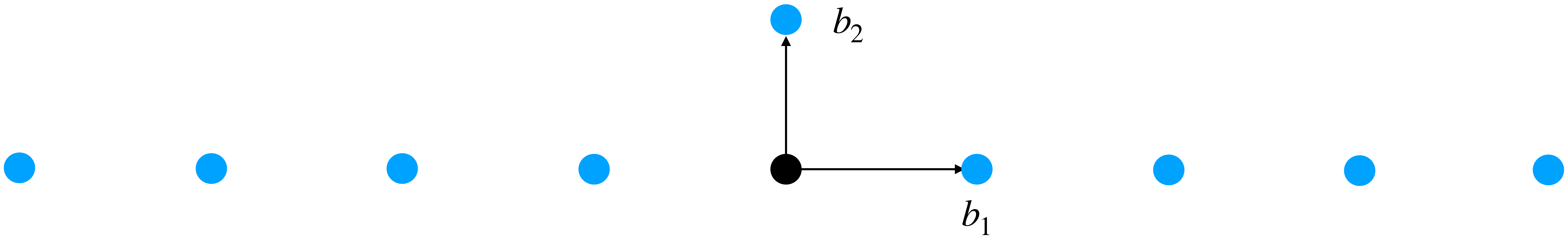

b_2
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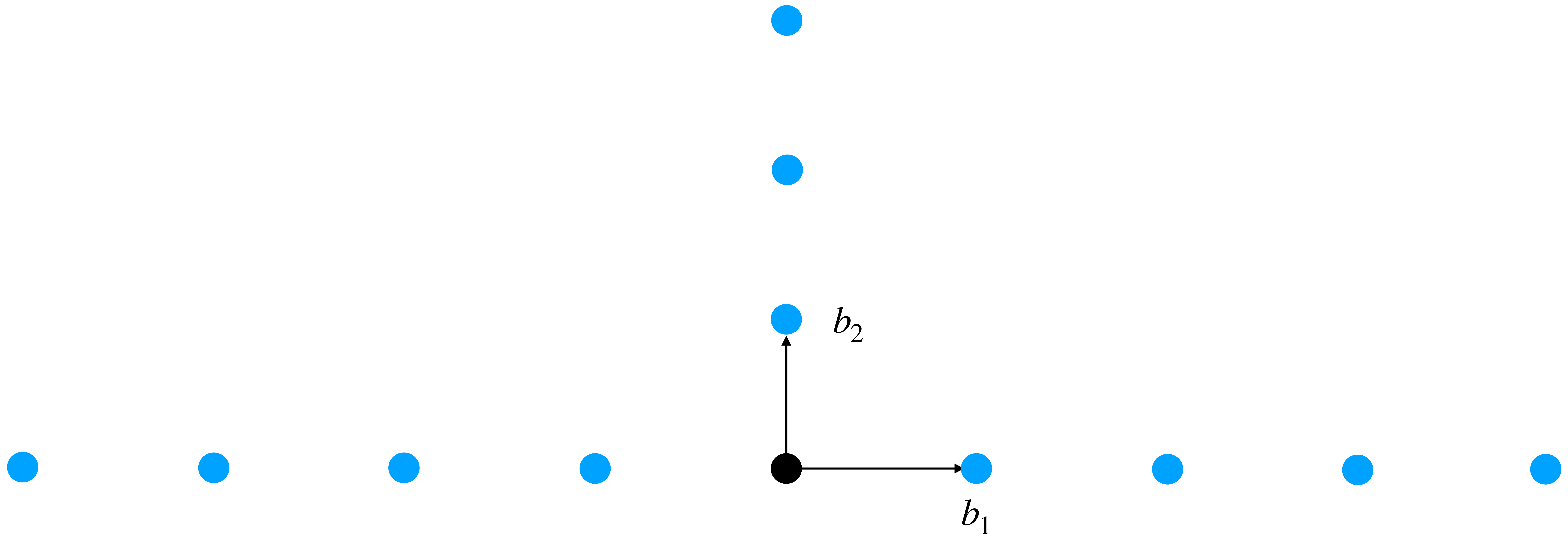
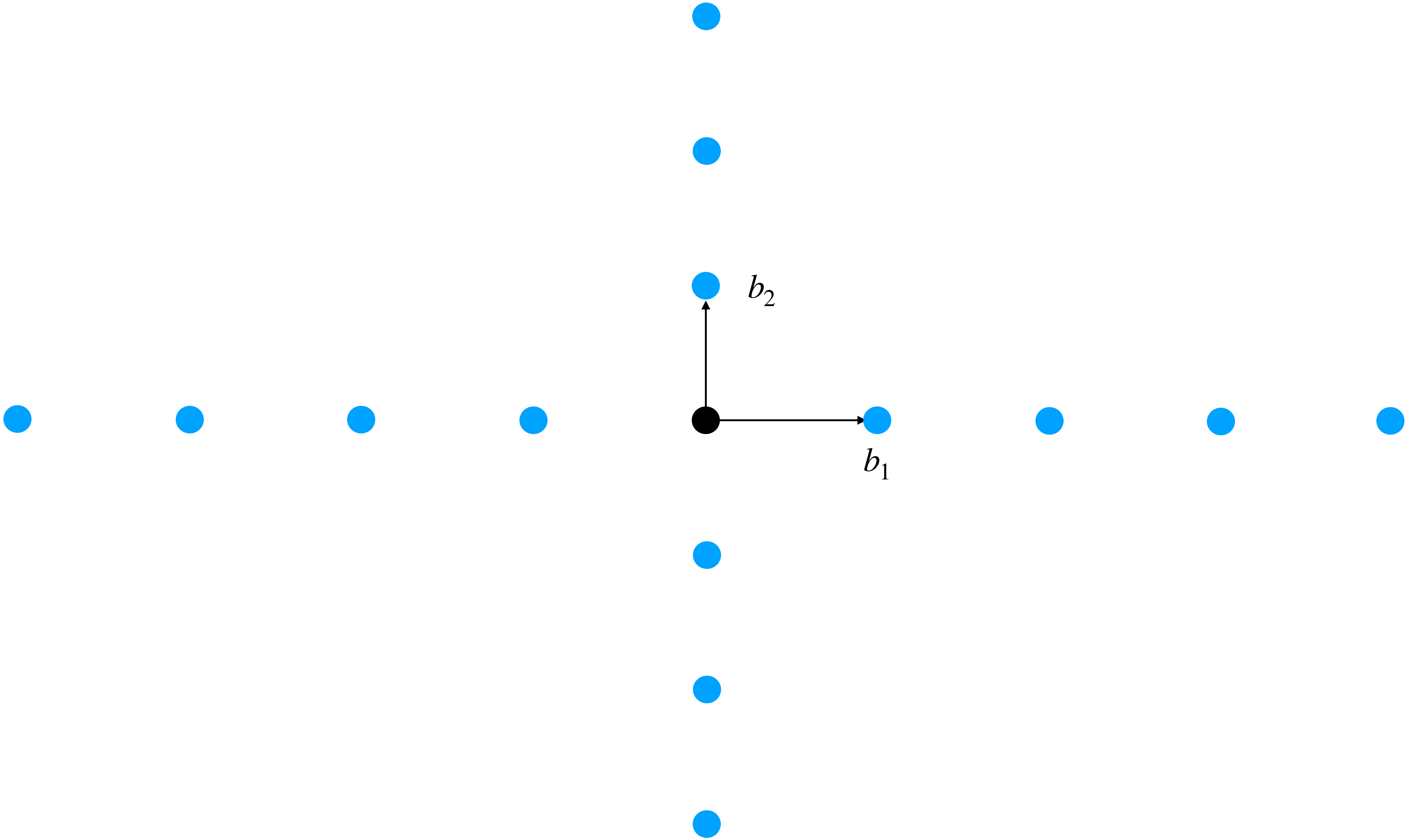


b_2
b_1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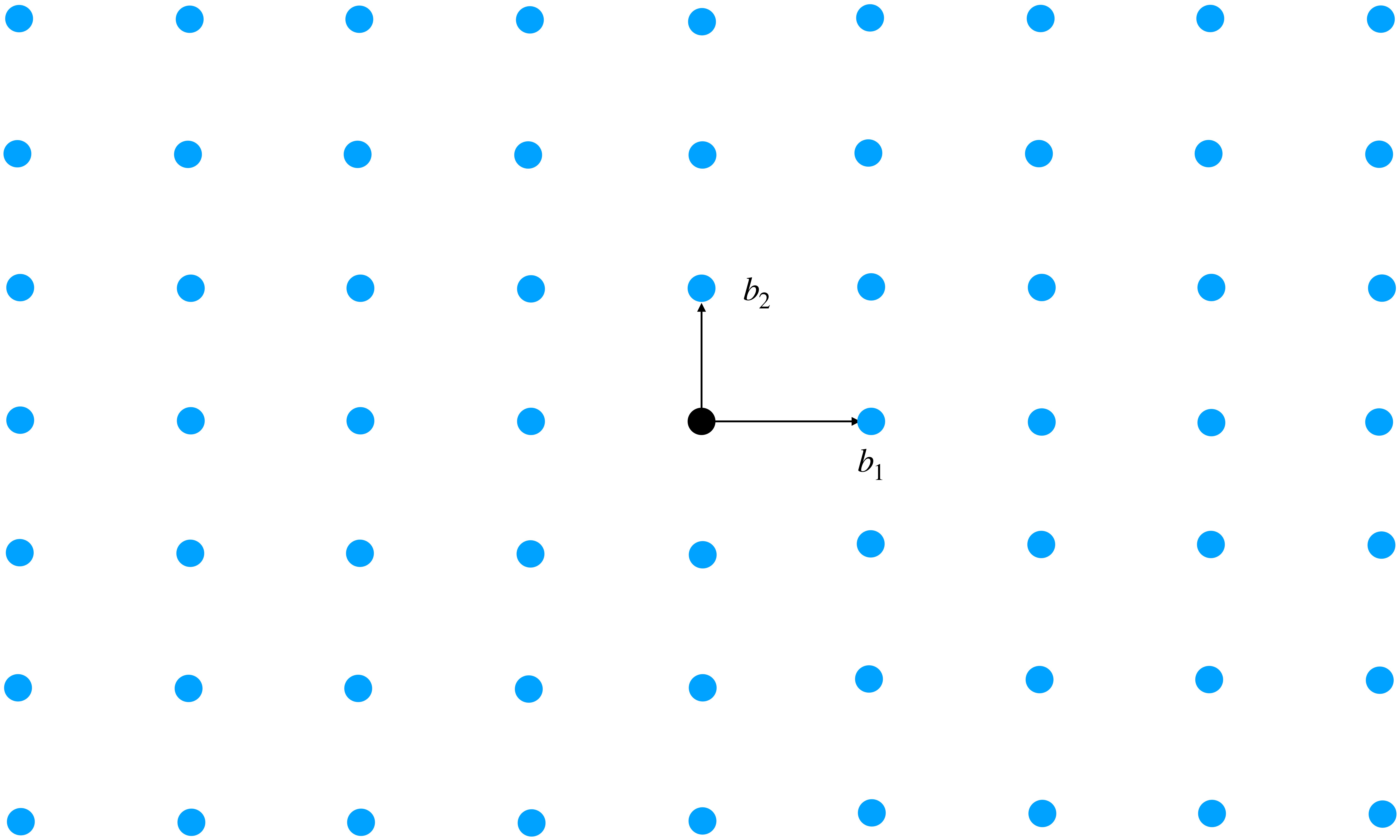
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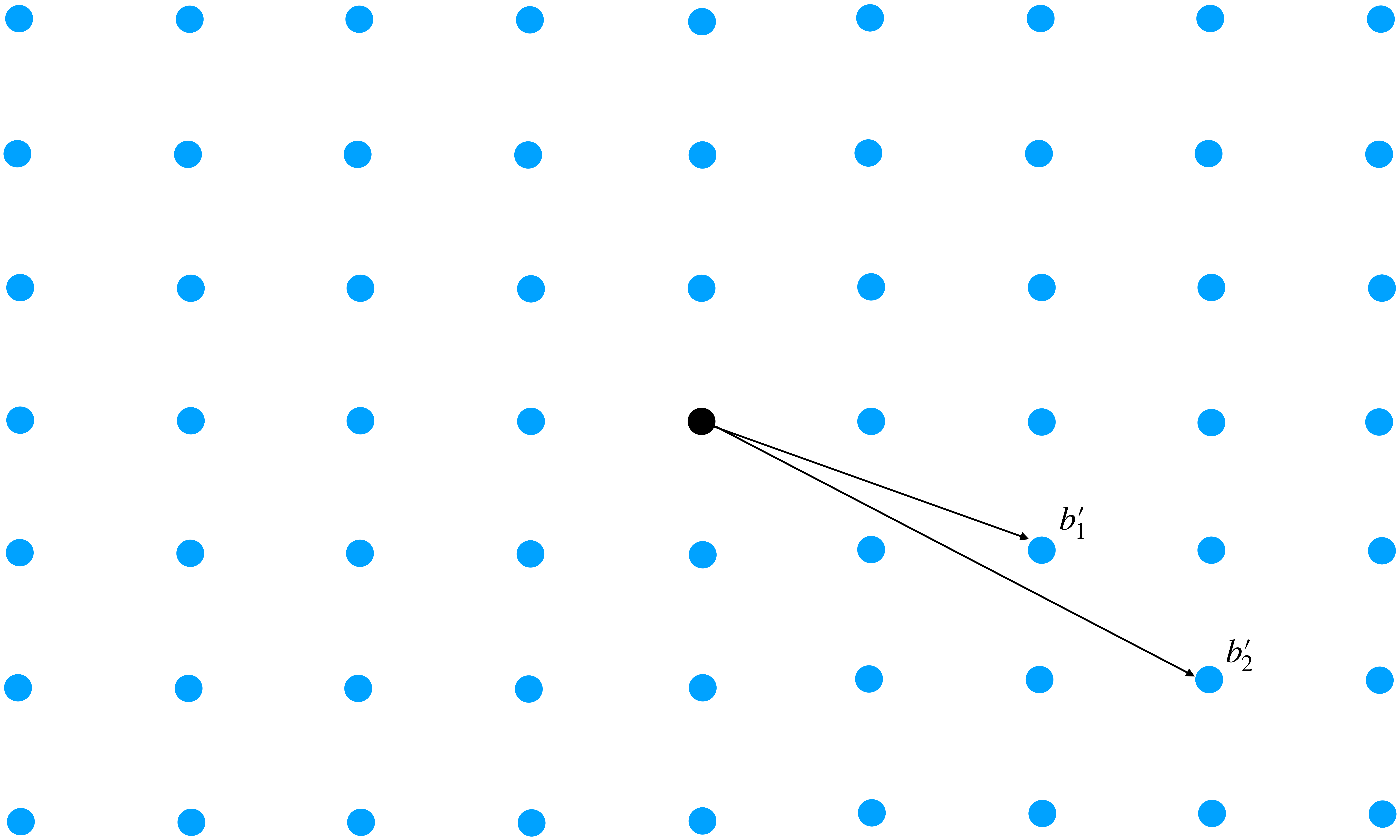

b_1'
b_2'

Closest Vector Problem (CVP)

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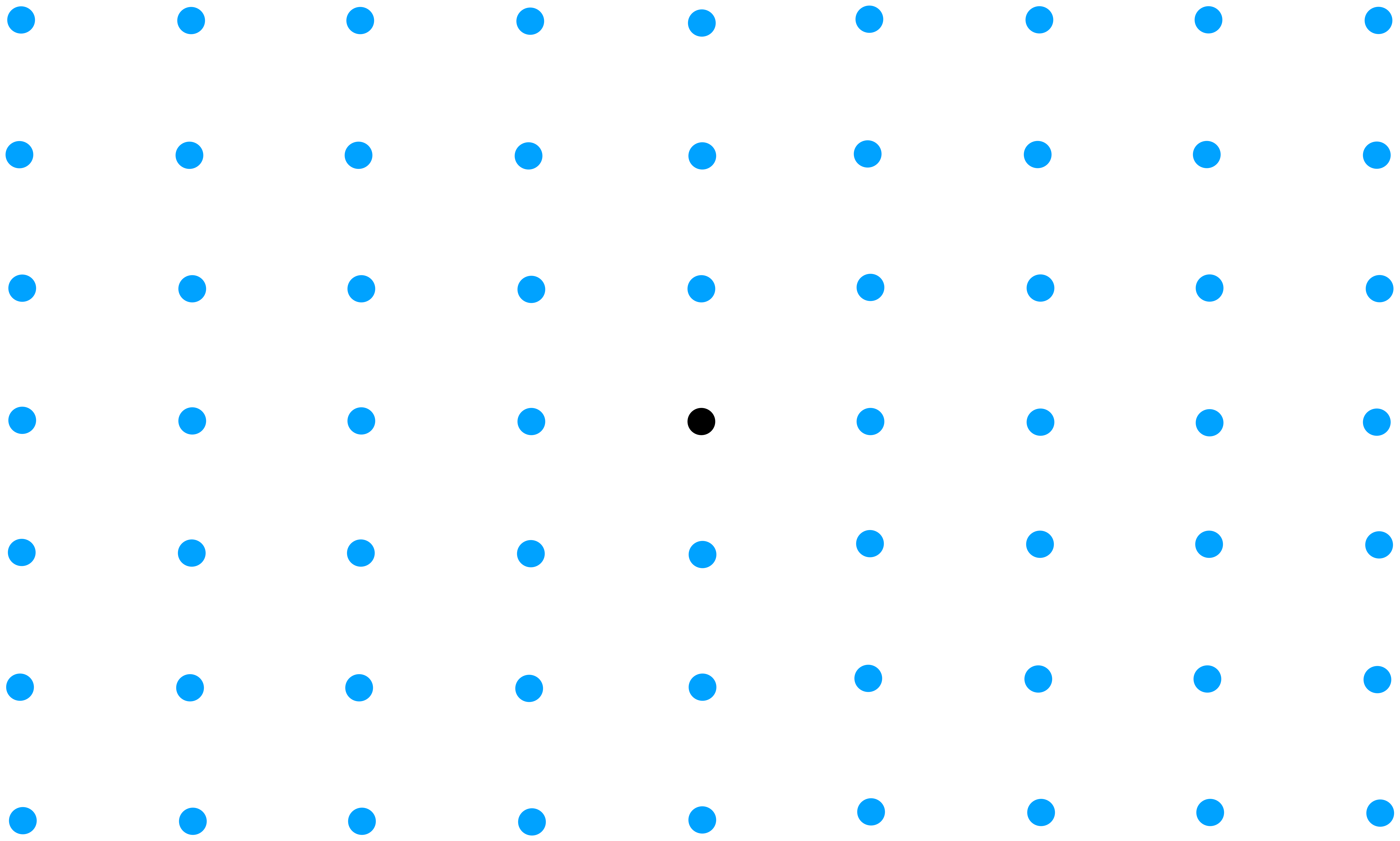
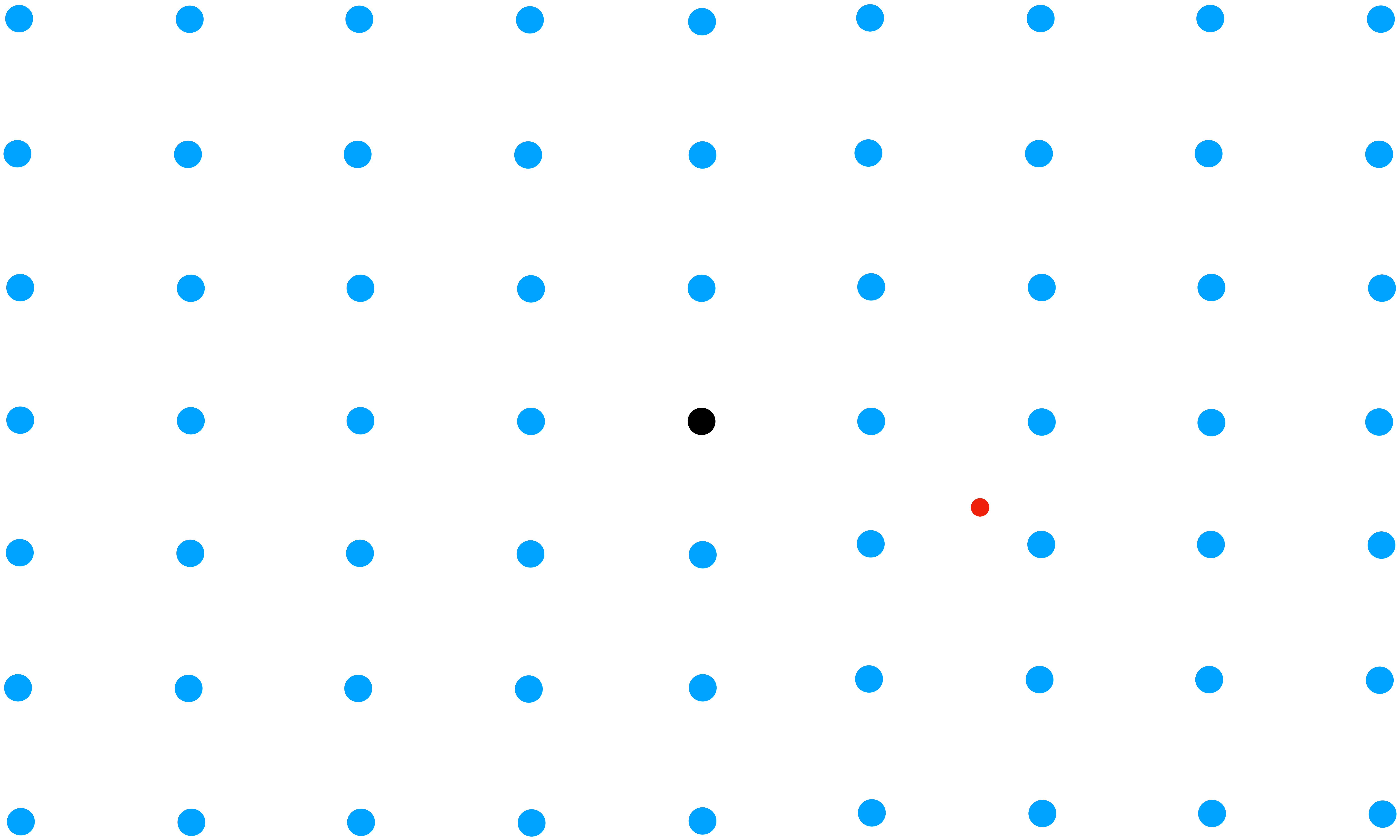
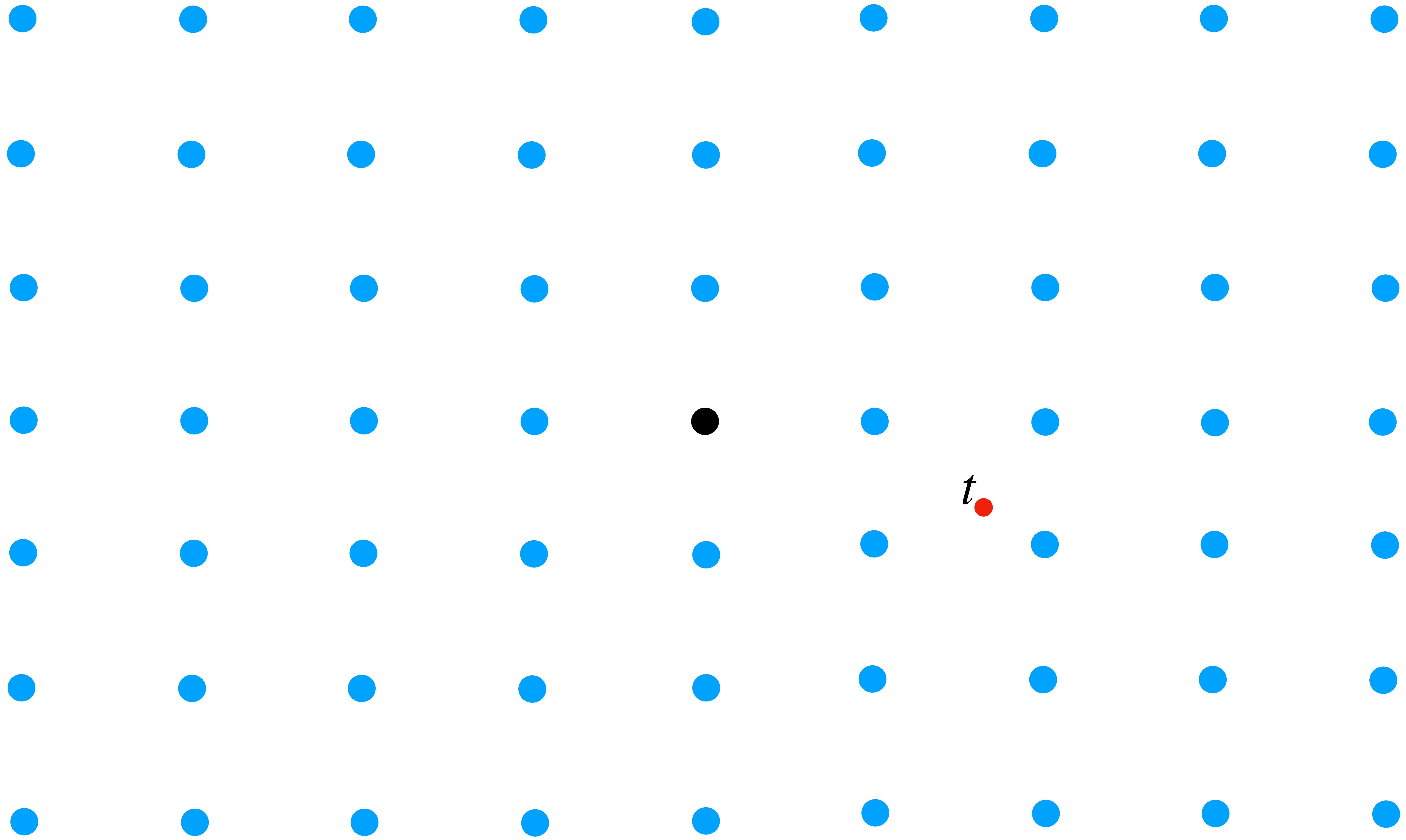
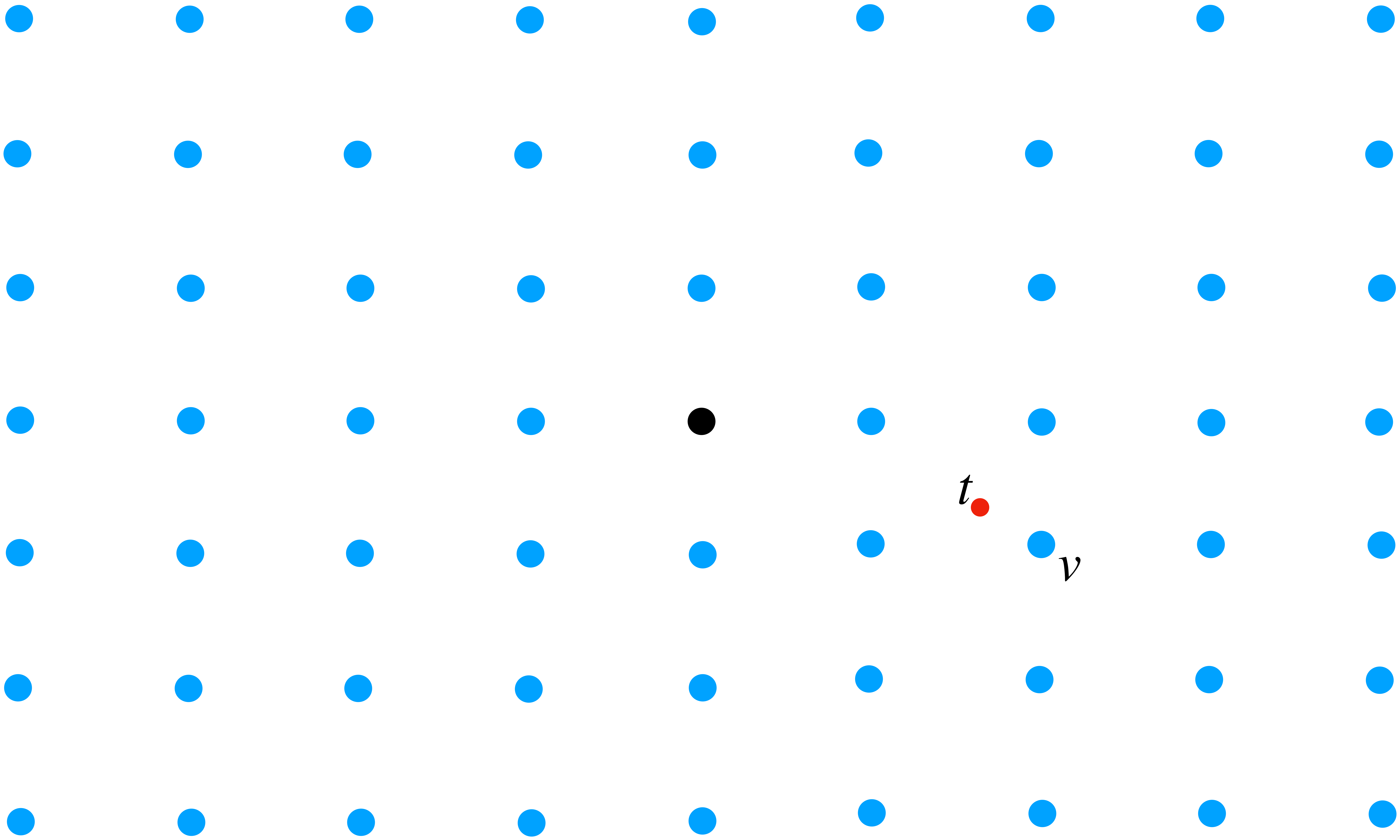
Given a basis $B = \{b_1, \ldots, b_n\}$ and a target $t \in \mathbb{R}^n$, find a vector $v \in \mathscr{L}(B)$ such that v is closest to t, i.e.,

$$||v - t|| \leq ||u - t||, \forall u \in \mathscr{L}(B)$$






t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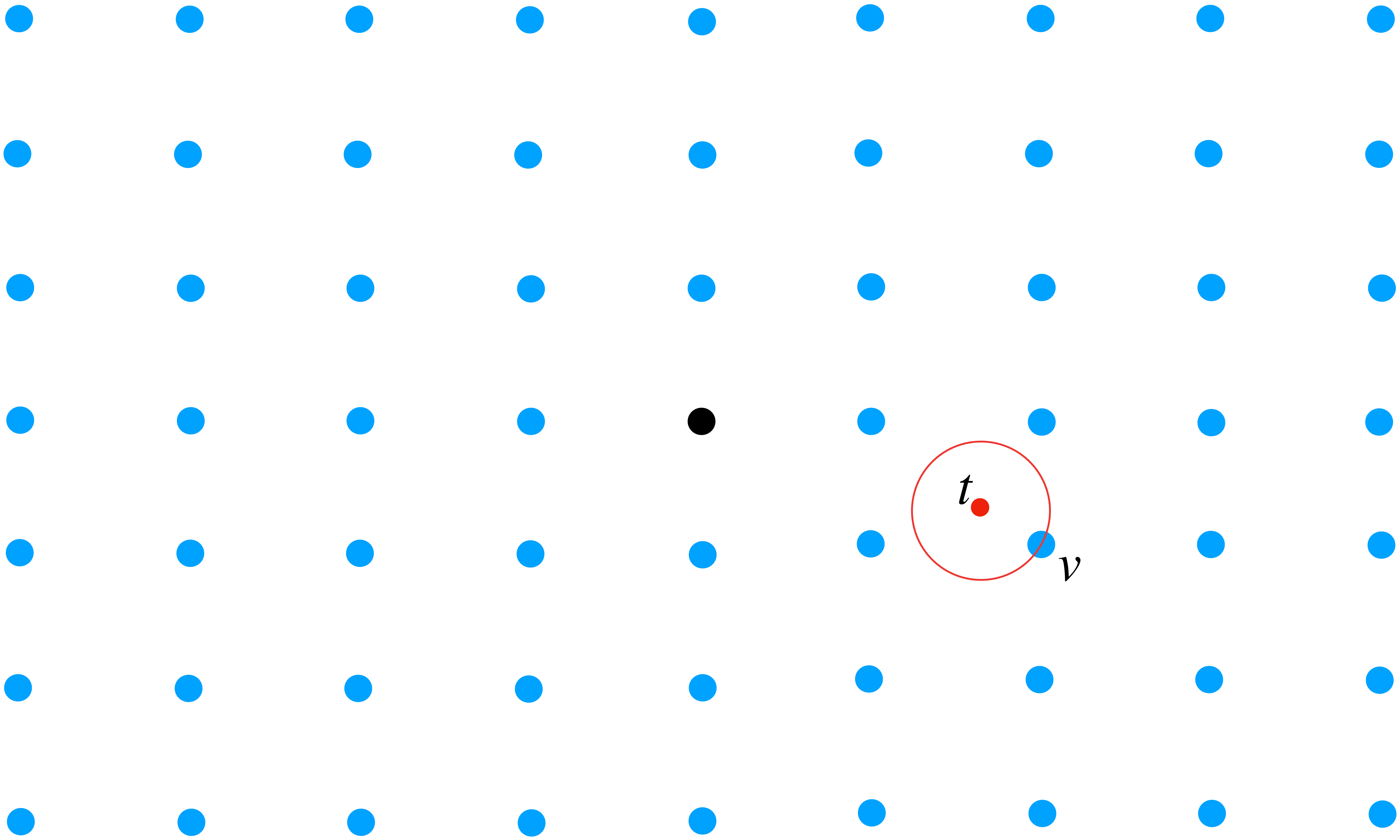
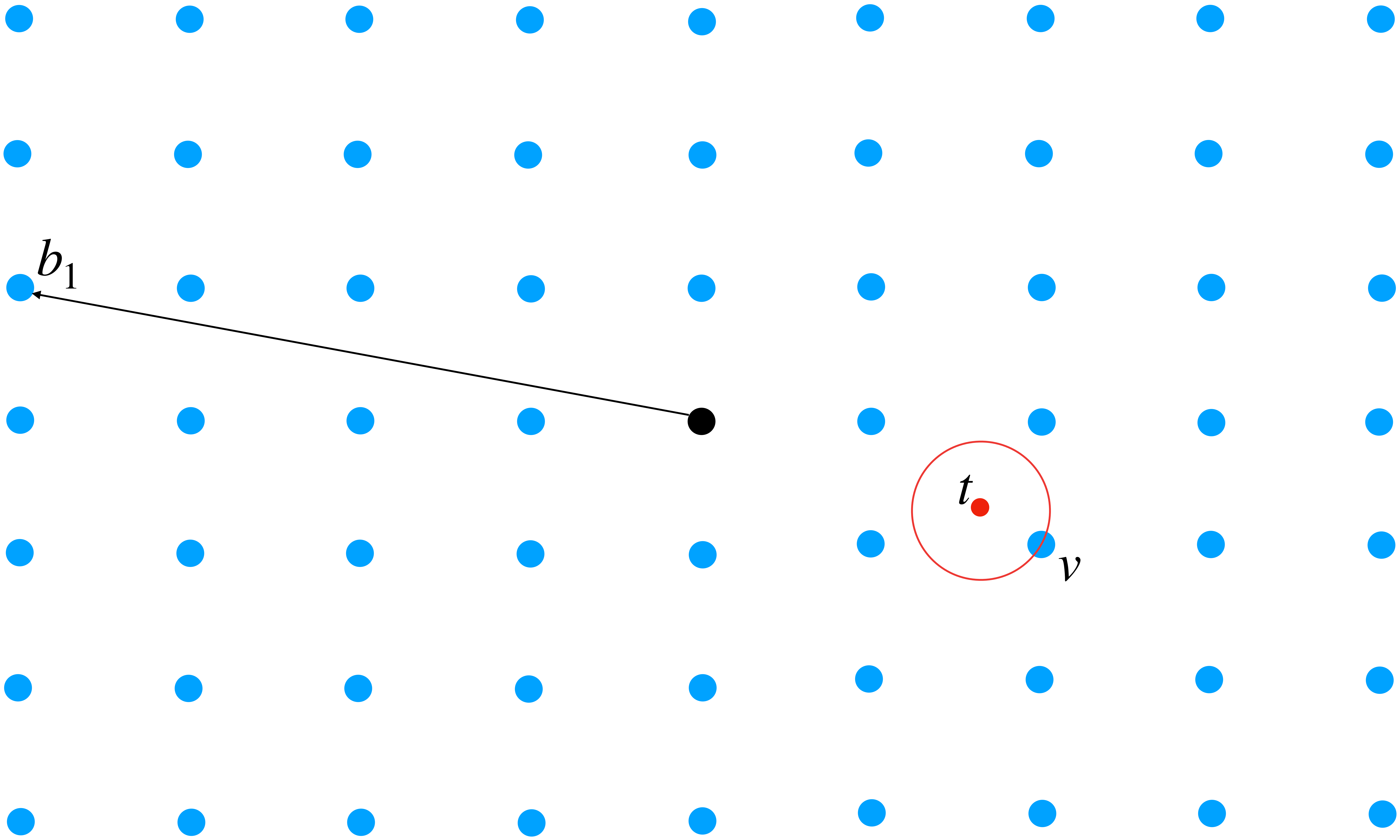
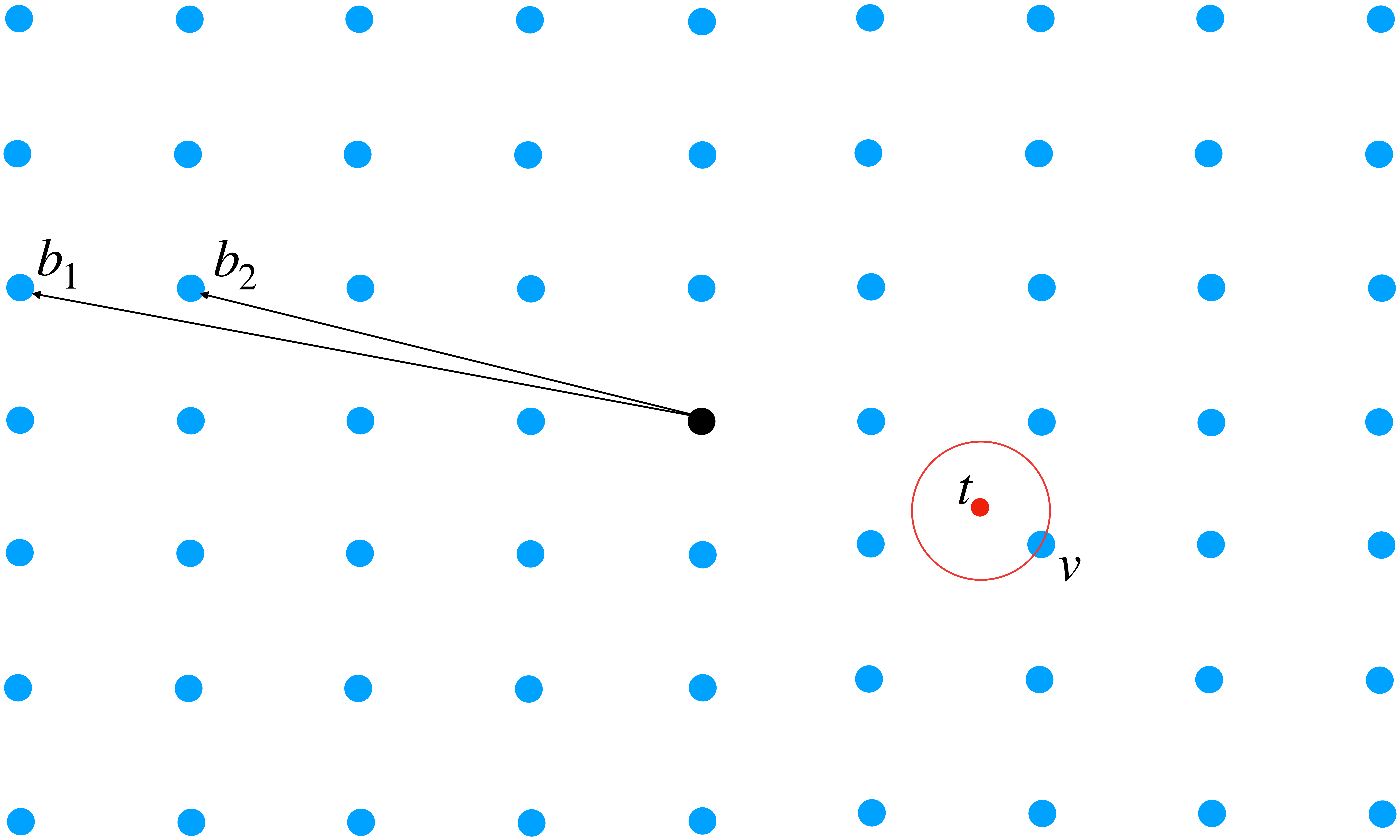

t
v


b_1
t
v


b_1
b_2
t
v

UBSSUM to CVP_∞

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Input: Unbounded subset sum instance $a_1, \ldots, a_n, t$

UBSSUM to CVP_∞

Input: Unbounded subset sum instance $a_1, \ldots, a_n, t$

$$B = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & & & \\ 0 & 0 & \cdots & 1 \\ \lambda a_1 & \lambda a_2 & \cdots & \lambda a_n \end{bmatrix}, t = \begin{bmatrix} b \\ b \\ \vdots \\ b \\ \lambda b \end{bmatrix}, d = b, t - Bx = \begin{bmatrix} b - x_1 \\ b - x_2 \\ \vdots \\ b - x_n \\ \lambda(b - \sum a_i x_i) \end{bmatrix}$$

UBSSUM to CVP_∞

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Assume, CVP is **YES**, then this implies that $\exists x$ such that $||t - Bx||_\infty \leq b$.

UBSSUM to CVP$_\infty$

Input: Unbounded subset sum instance $a_1, \ldots, a_n, t$

$$B = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & & & \\ 0 & 0 & \cdots & 1 \\ \lambda a_1 & \lambda a_2 & \cdots & \lambda a_n \end{bmatrix}, t = \begin{bmatrix} b \\ b \\ \vdots \\ b \\ \lambda b \end{bmatrix}, d = b, t - Bx = \begin{bmatrix} b - x_1 \\ b - x_2 \\ \vdots \\ b - x_n \\ \lambda(b - \sum a_i x_i) \end{bmatrix}$$

Assume, CVP is **YES**, then this implies that $\exists x$ such that $||t - Bx||_\infty \leq b$.

Since, λ is very large, this implies $(t - Bx)_{n+1} = 0 \implies \vec{a} \cdot x = b$. Also, $x_i \geq 0$.

UBSSUM to CVP_1

UBSSUM to CVP_1

Input: Unbounded subset sum instance $a_1, \ldots, a_n, t$

UBSSUM to CVP_1

Input: Unbounded subset sum instance $a_1, \ldots, a_n, t$

$$B = \begin{bmatrix} -a_1 & 0 & \cdots & 0 \\ 0 & -a_2 & \cdots & 0 \\ \vdots & & & \\ 0 & 0 & \cdots & -a_n \\ \lambda a_1 & \lambda a_2 & \cdots & \lambda a_n \end{bmatrix}, t = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ \lambda b \end{bmatrix}, d = b, t - Bx = \begin{bmatrix} a_1 x_1 \\ a_2 x_2 \\ \vdots \\ a_n x_n \\ \lambda(b - \sum a_i x_i) \end{bmatrix}$$

UBSSUM to CVP_1

Input: Unbounded subset sum instance $a_1, \ldots, a_n, t$

$$B = \begin{bmatrix} -a_1 & 0 & \cdots & 0 \\ 0 & -a_2 & \cdots & 0 \\ \vdots & & & \\ 0 & 0 & \cdots & -a_n \\ \lambda a_1 & \lambda a_2 & \cdots & \lambda a_n \end{bmatrix}, t = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ \lambda b \end{bmatrix}, d = b, t - Bx = \begin{bmatrix} a_1 x_1 \\ a_2 x_2 \\ \vdots \\ a_n x_n \\ \lambda(b - \sum a_i x_i) \end{bmatrix}$$

Assume, CVP is **YES**, then this implies that $\exists x$ such that $||t - Bx||_1 \leq b$.

UBSSUM to CVP_1

Input: Unbounded subset sum instance $a_1, \ldots, a_n, t$

$$B = \begin{bmatrix} -a_1 & 0 & \cdots & 0 \\ 0 & -a_2 & \cdots & 0 \\ \vdots & & & \\ 0 & 0 & \cdots & -a_n \\ \lambda a_1 & \lambda a_2 & \cdots & \lambda a_n \end{bmatrix}, t = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ \lambda b \end{bmatrix}, d = b, t - Bx = \begin{bmatrix} a_1 x_1 \\ a_2 x_2 \\ \vdots \\ a_n x_n \\ \lambda(b - \sum a_i x_i) \end{bmatrix}$$

Assume, CVP is **YES**, then this implies that $\exists x$ such that $||t - Bx||_1 \leq b$.

Since, λ is very large, this implies $(t - Bx)_{n+1} = 0 \implies \vec{a} \cdot x = b$. Also,

UBSSUM to CVP$_1$

Input: Unbounded subset sum instance $a_1, \ldots, a_n, t$

$$B = \begin{bmatrix} -a_1 & 0 & \cdots & 0 \\ 0 & -a_2 & \cdots & 0 \\ \vdots & & & \\ 0 & 0 & \cdots & -a_n \\ \lambda a_1 & \lambda a_2 & \cdots & \lambda a_n \end{bmatrix}, t = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ \lambda b \end{bmatrix}, d = b, t - Bx = \begin{bmatrix} a_1 x_1 \\ a_2 x_2 \\ \vdots \\ a_n x_n \\ \lambda(b - \sum a_i x_i) \end{bmatrix}$$

Assume, CVP is **YES**, then this implies that $\exists x$ such that $||t - Bx||_1 \leq b$.

Since, λ is very large, this implies $(t - Bx)_{n+1} = 0 \implies \vec{a} \cdot x = b$. Also,

$$b \geq \sum_{i=1}^{n} |a_i x_i| = \sum_{i=1}^{n} a_i |x_i| \geq \sum_{i=1}^{n} a_i x_i = b$$

Other Results

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- There is an $\tilde{O}(k(n+t))$ time deterministic algorithm for Hamming-k-SUBSSUM (ask to return all $\sum_{i\in[n]} \beta_i$ where the number of solutions is atmost k)

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- There is an $\tilde{O}(k(n+t))$ time deterministic algorithm for Hamming-k-SUBSSUM (ask to return all $\sum_{i\in[n]} \beta_i$ where the number of solutions is atmost k)
- There is a $poly(knt)$-time and $O(log(knt))$-space deterministic algorithm which solves k-SUBSSUM (ask to return all $(\beta_1, \ldots, \beta_n)$ where the number of solutions is atmost k)

Conclusion

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Conclusion

- We saw an $O(nt)$ and $\tilde{O}(n+t)$ time algorithm for unique-PSSUM_1.
- We saw reductions from UBSSUM to CVP.
- Can we find an $\tilde{O}(n+t)$ time algorithm for PSSUM_1.
- Extend it to other values of p?

Thank You!