On the bases of $\mathbb{Z}^n$ lattice

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Introduction

Lattice

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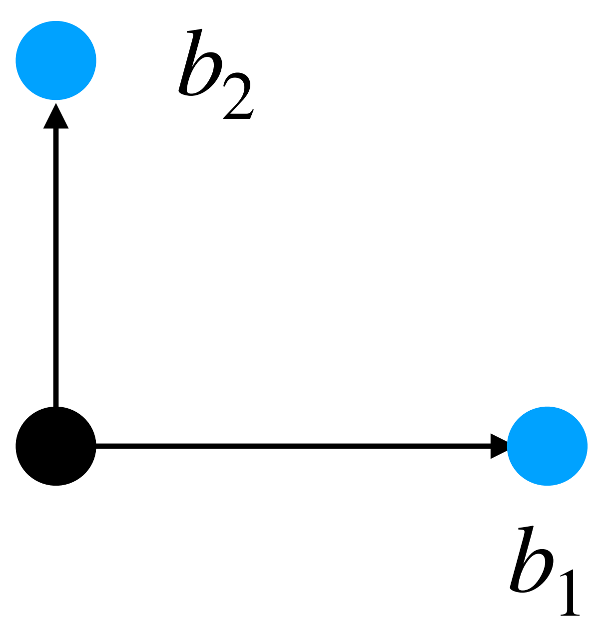
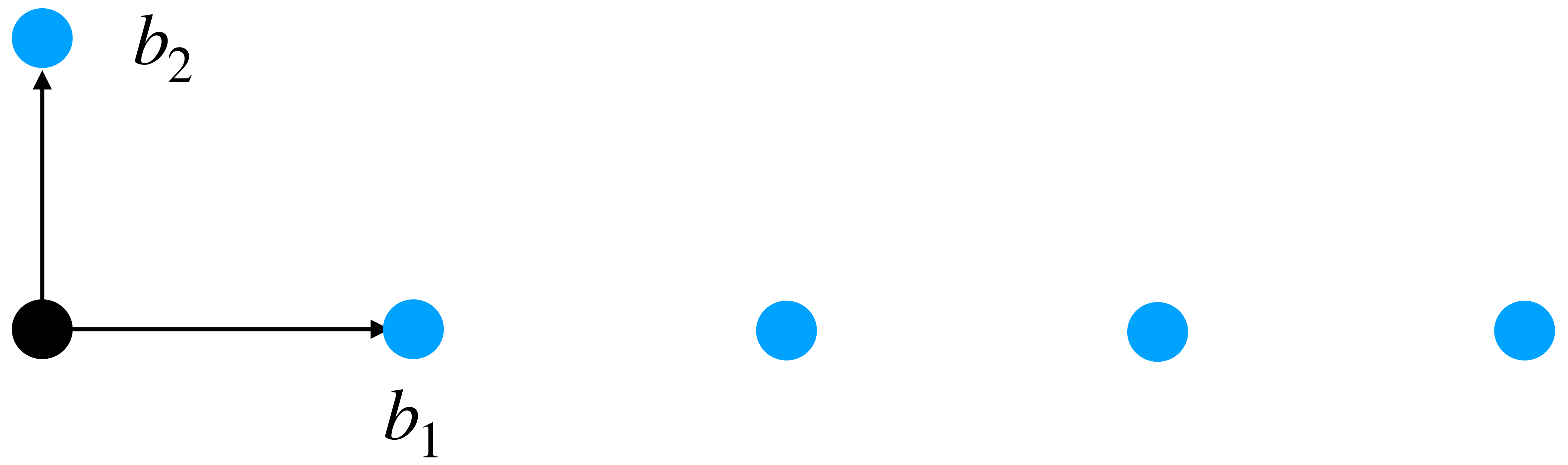
B is called a *basis* of $\mathscr{L}$.



b_2
b_1


b_2
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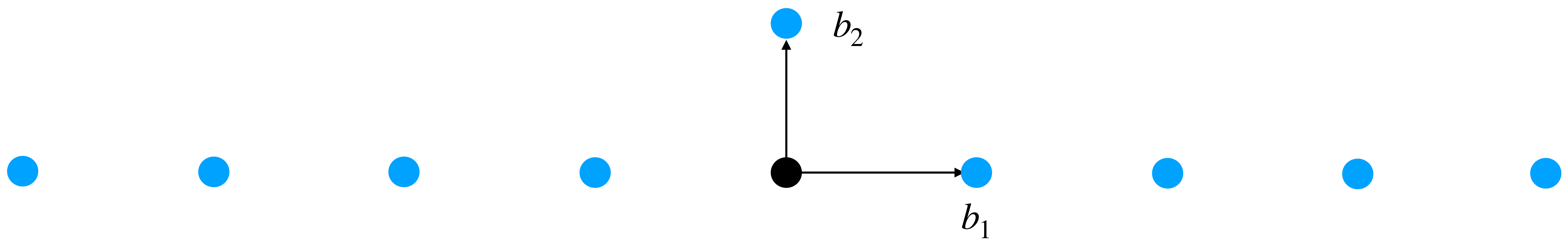
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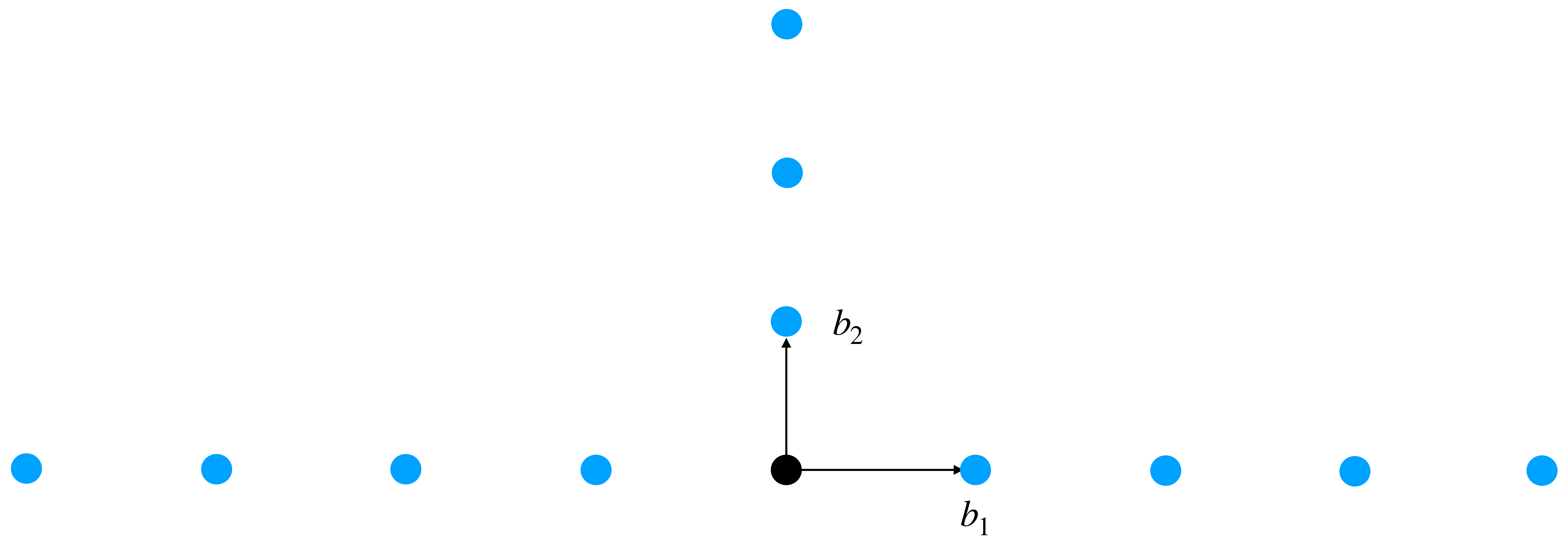
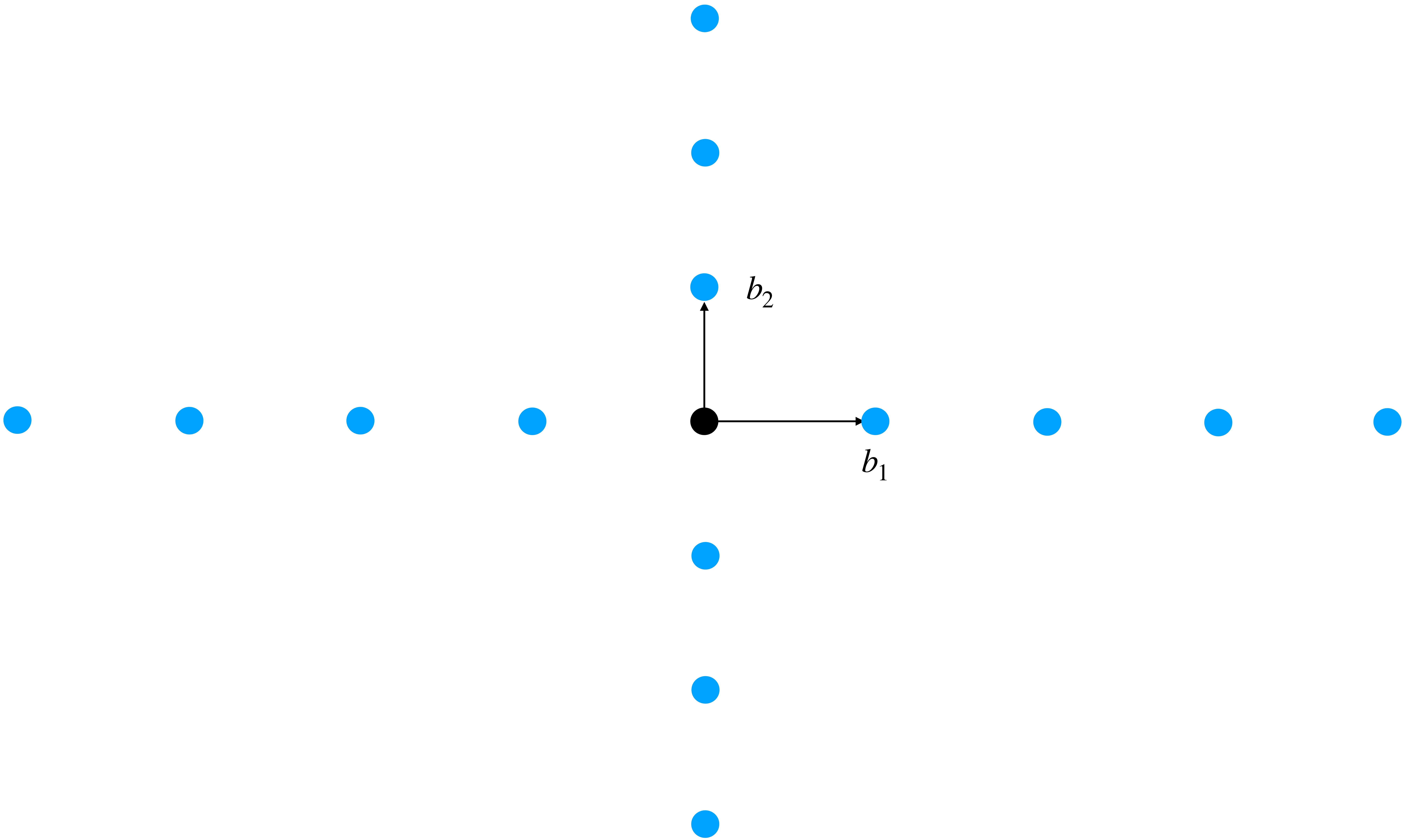
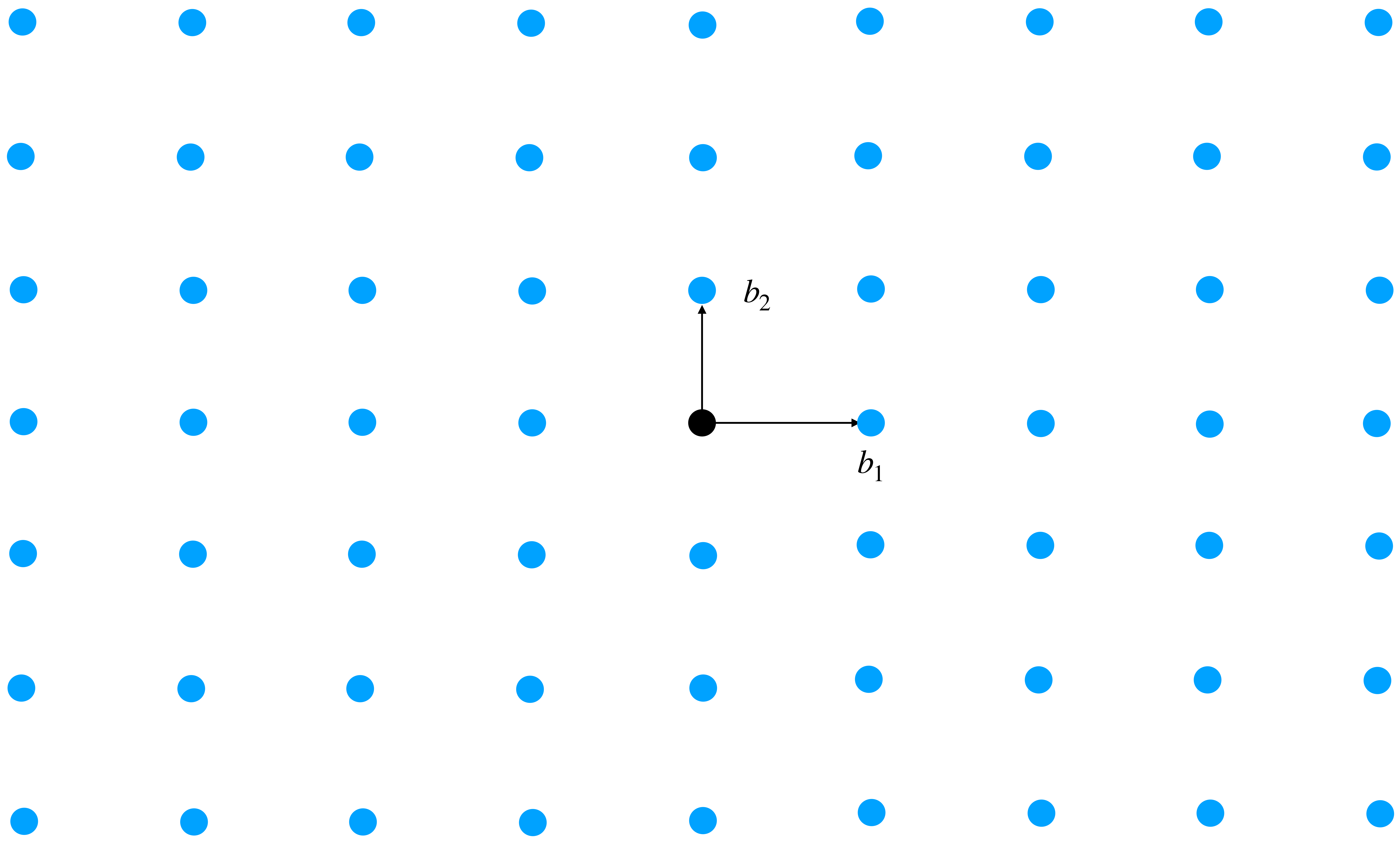
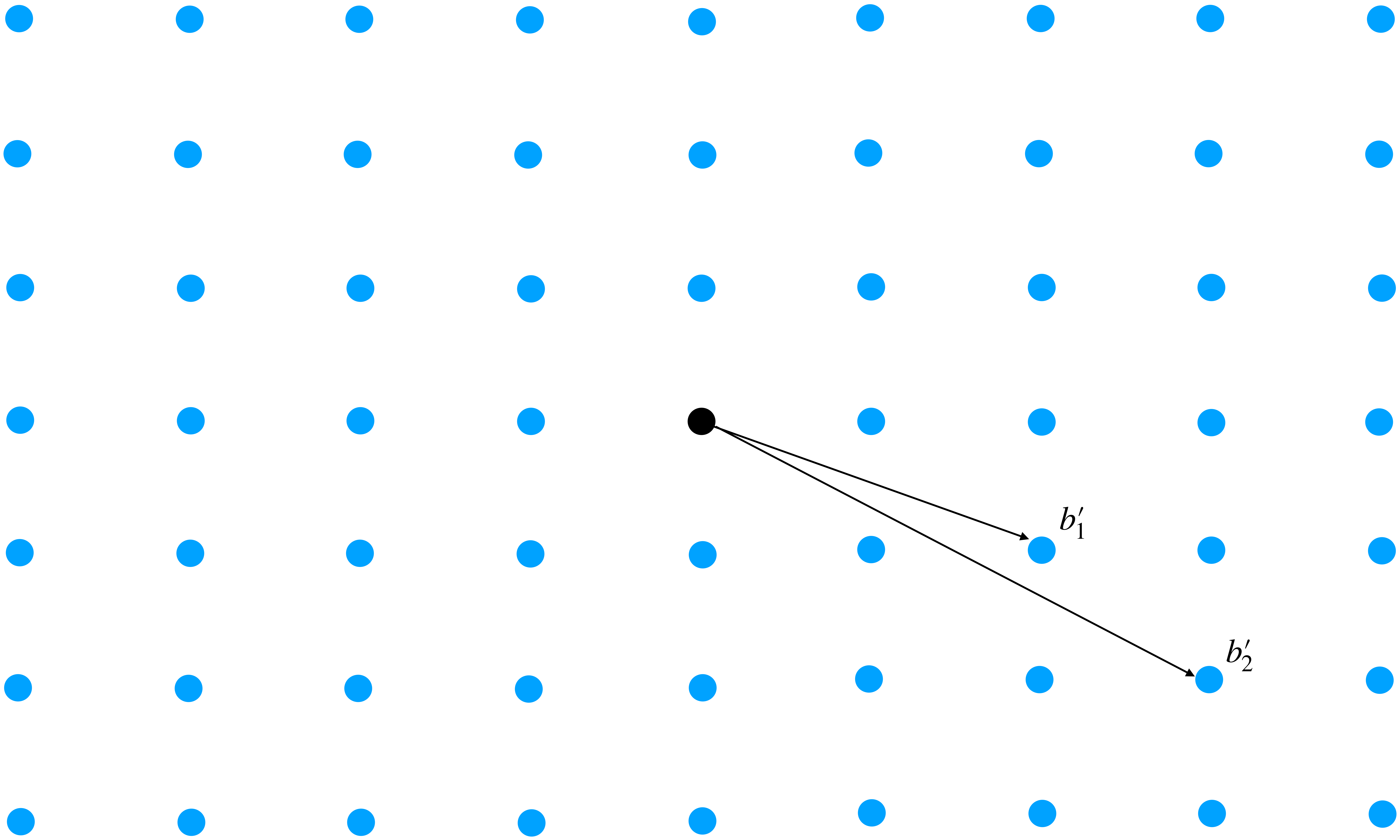


b_2
b_1


b_2
b_1



b_2
b_1


b'_1
b'_2

Bases of a lattice

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Therefore, a lattice can have infinitely many bases!

Applications

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- Building very strong cryptographic primitives (post-quantum).

Closest Vector Problem (CVP)

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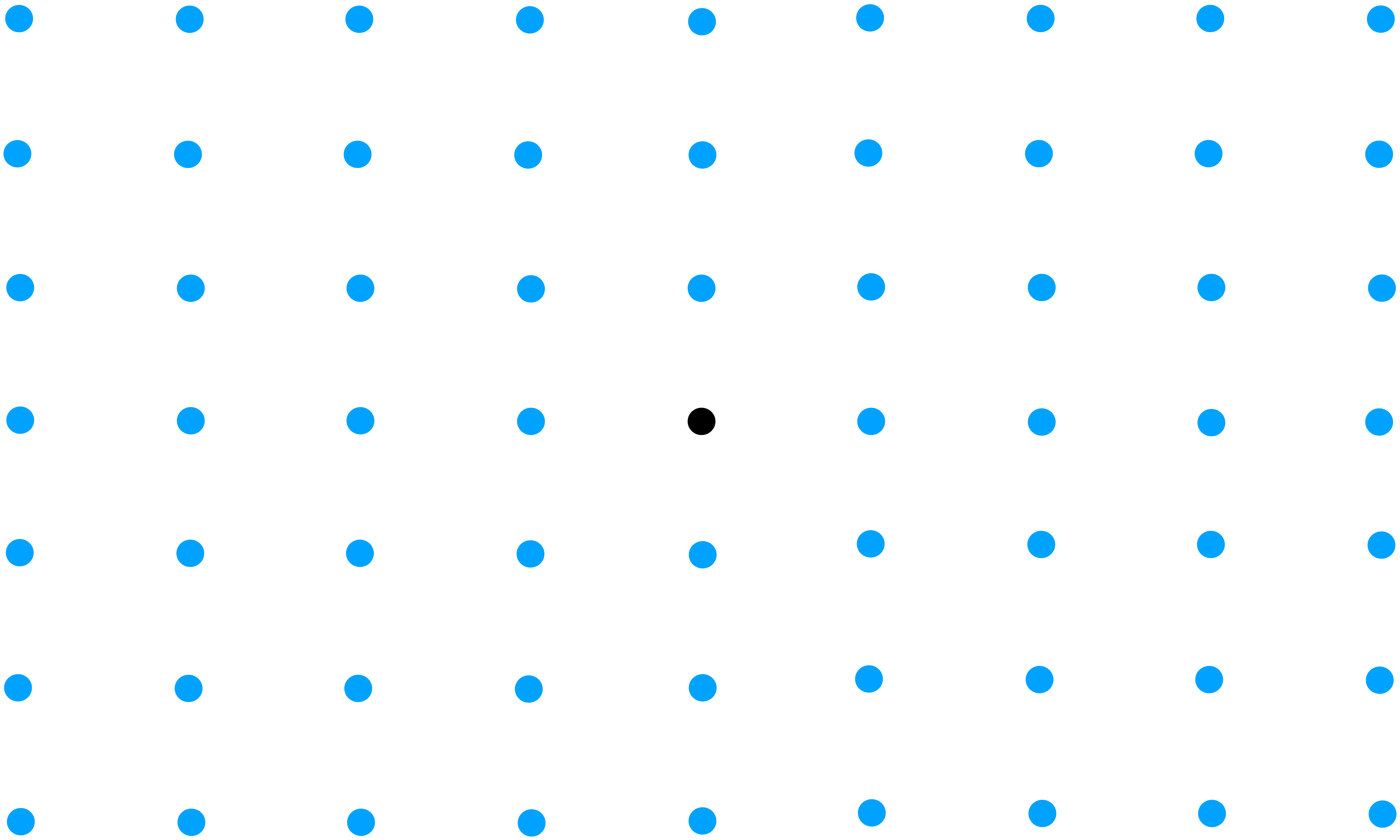
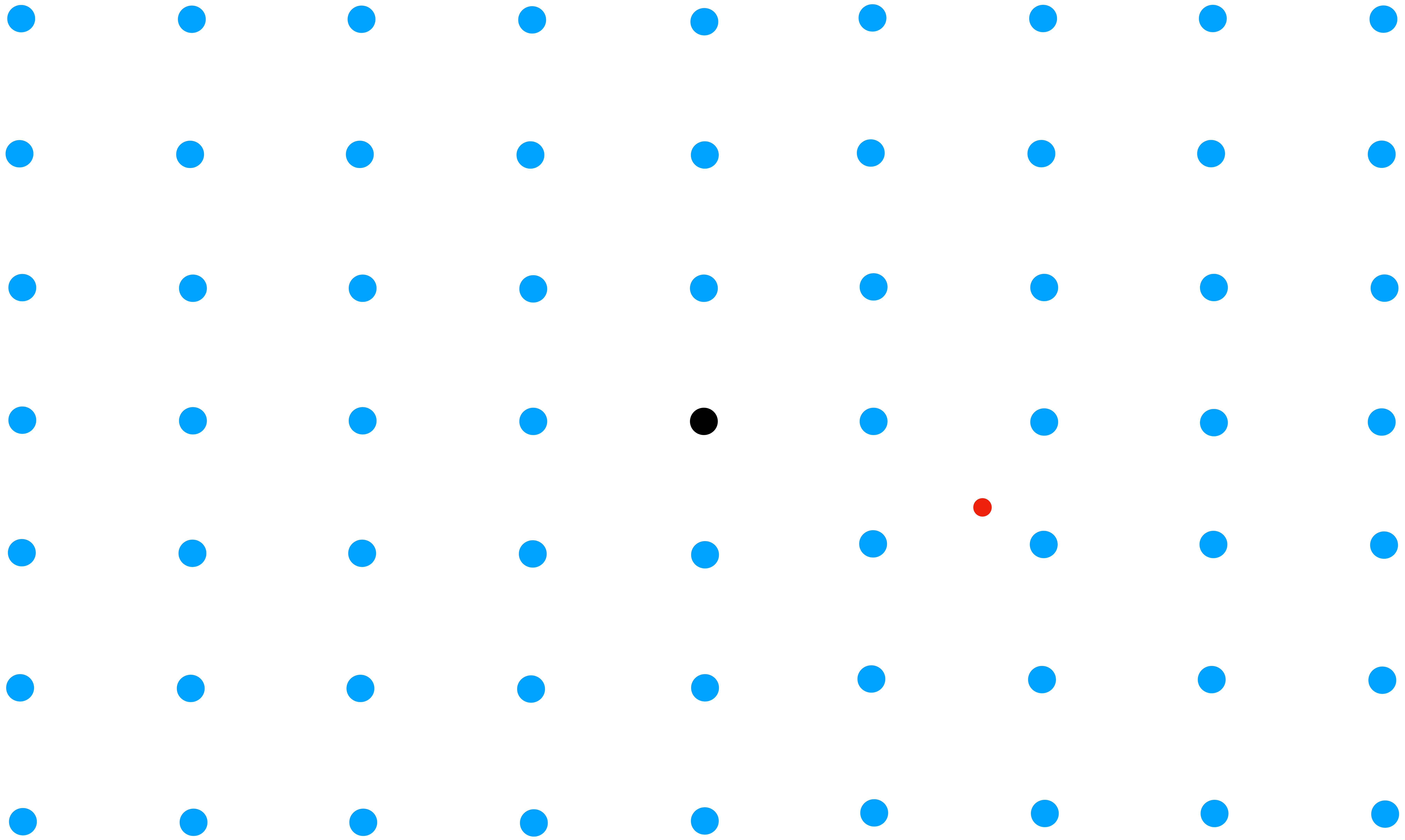
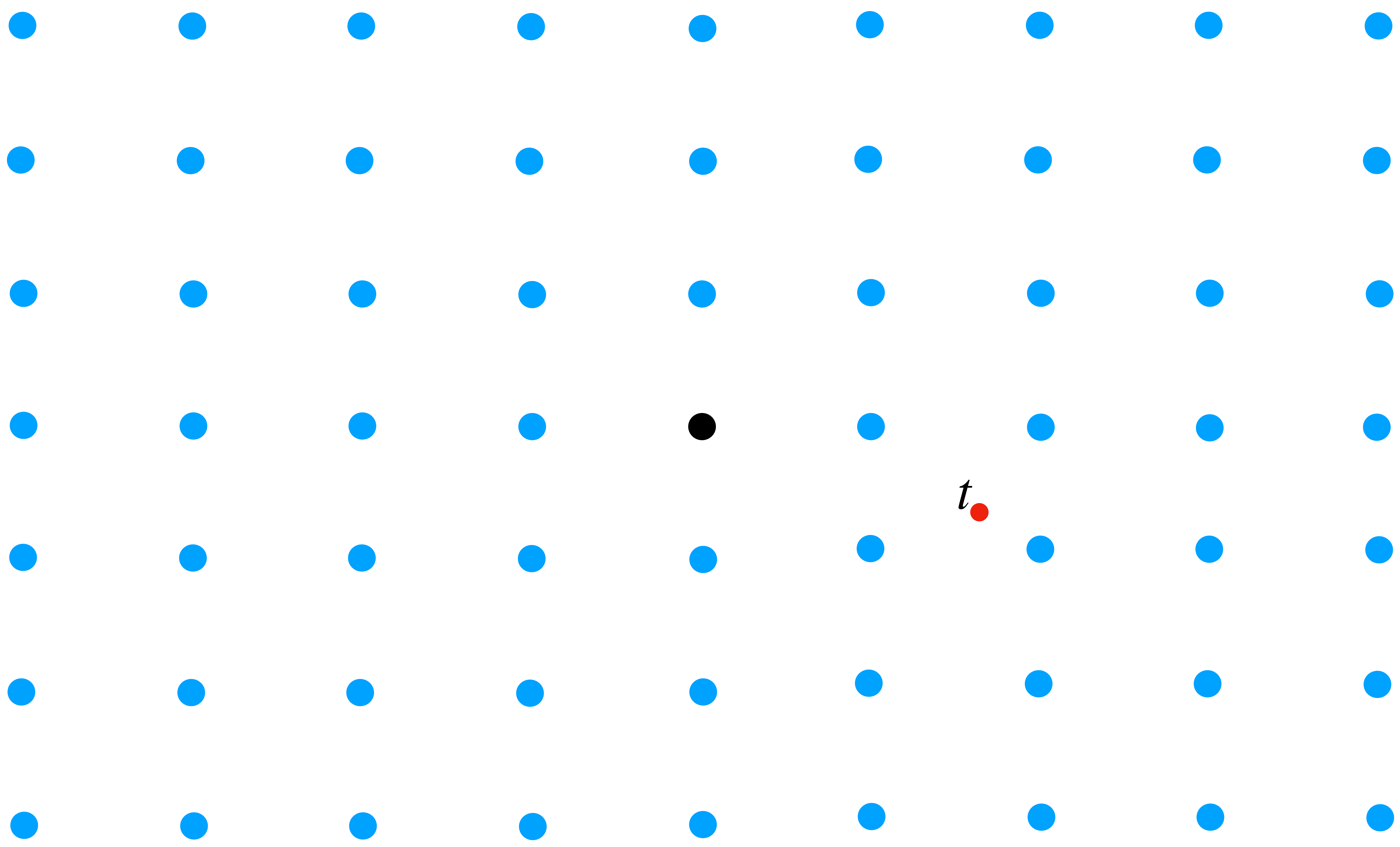
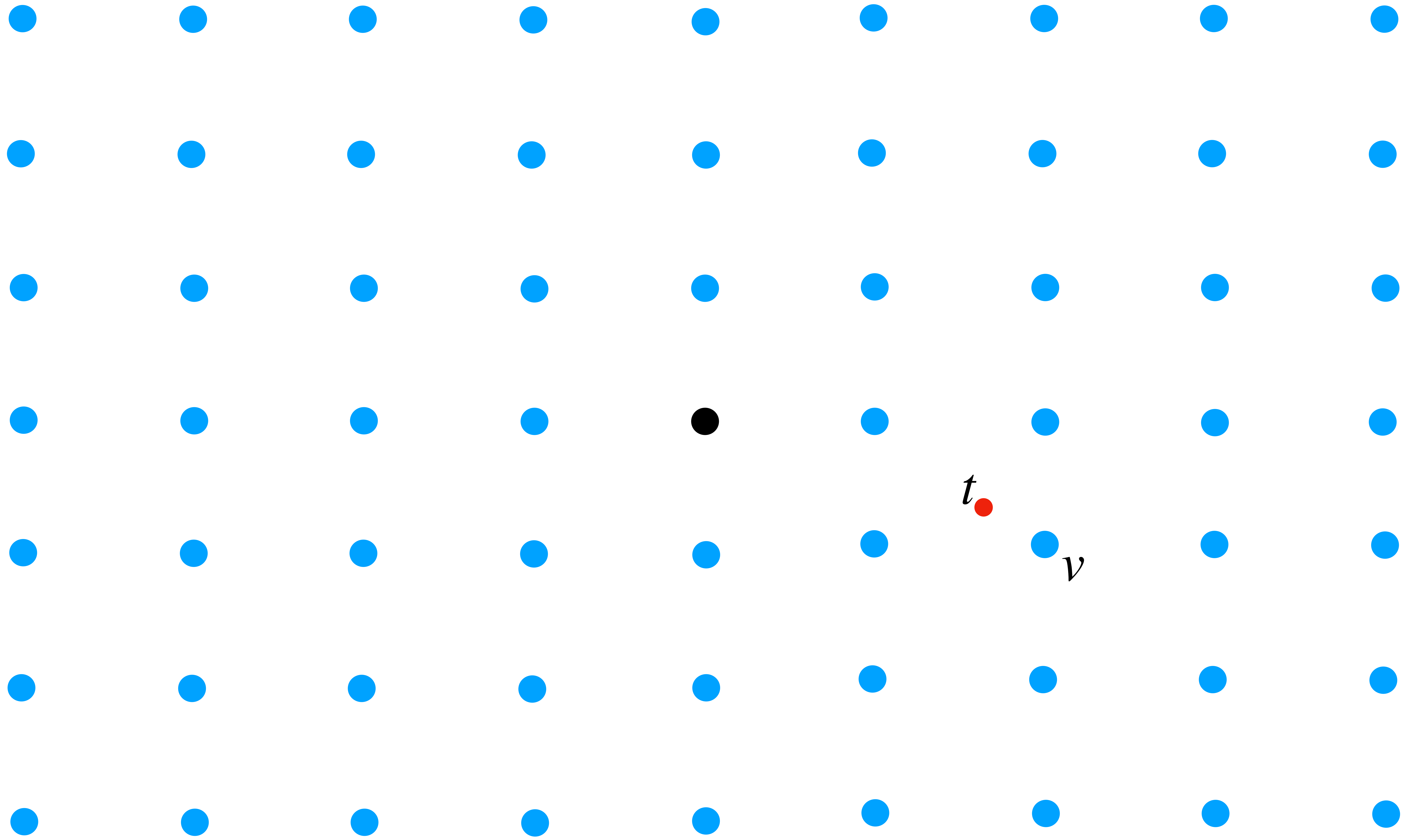
Given a basis $B = \{b_1, \ldots, b_n\}$ and a target $t \in \mathbb{R}^n$, find a vector $v \in \mathscr{L}(B)$ such that v is closest to t, i.e.,

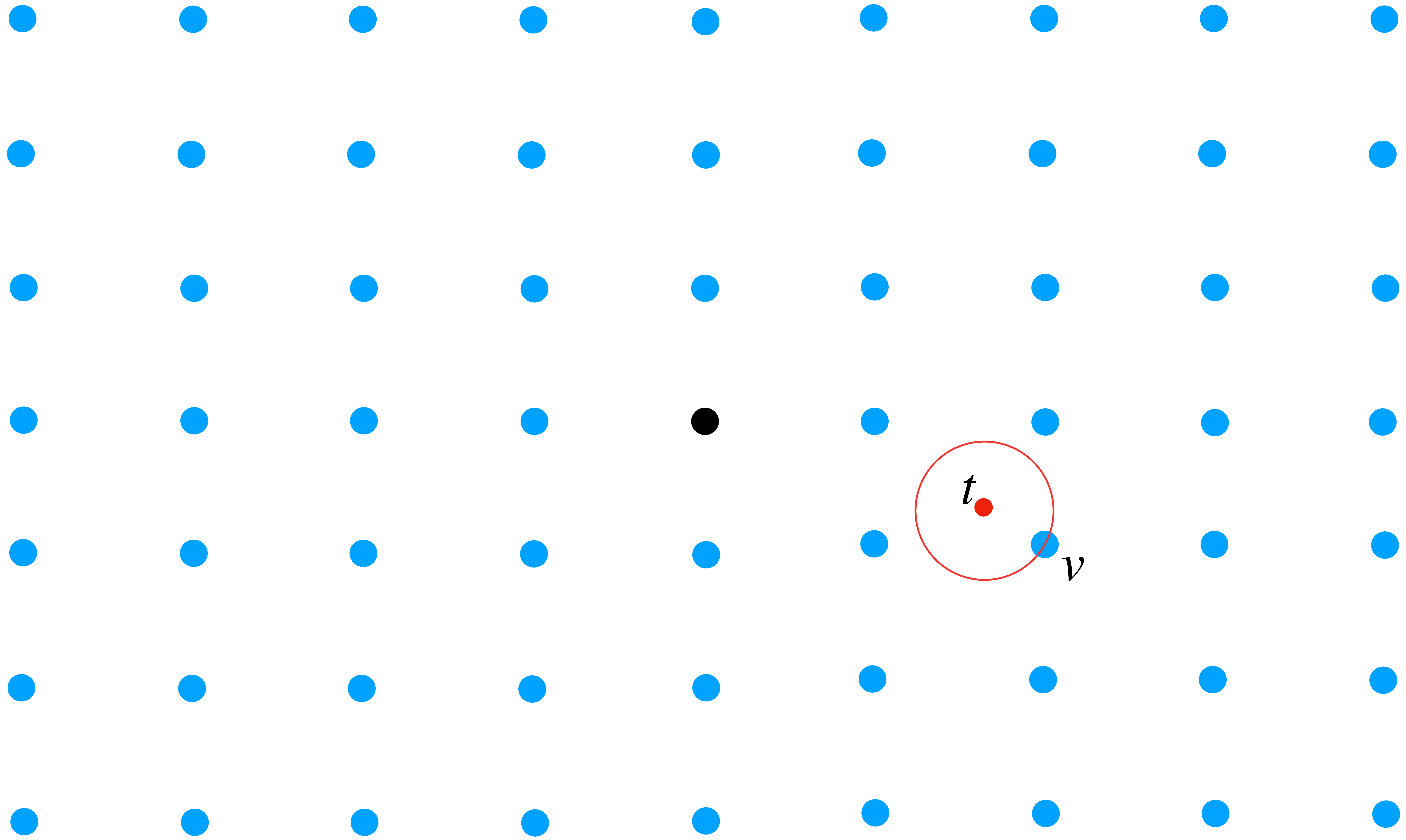
$$||v - t|| \leq ||u - t||, \forall u \in \mathscr{L}(B)$$



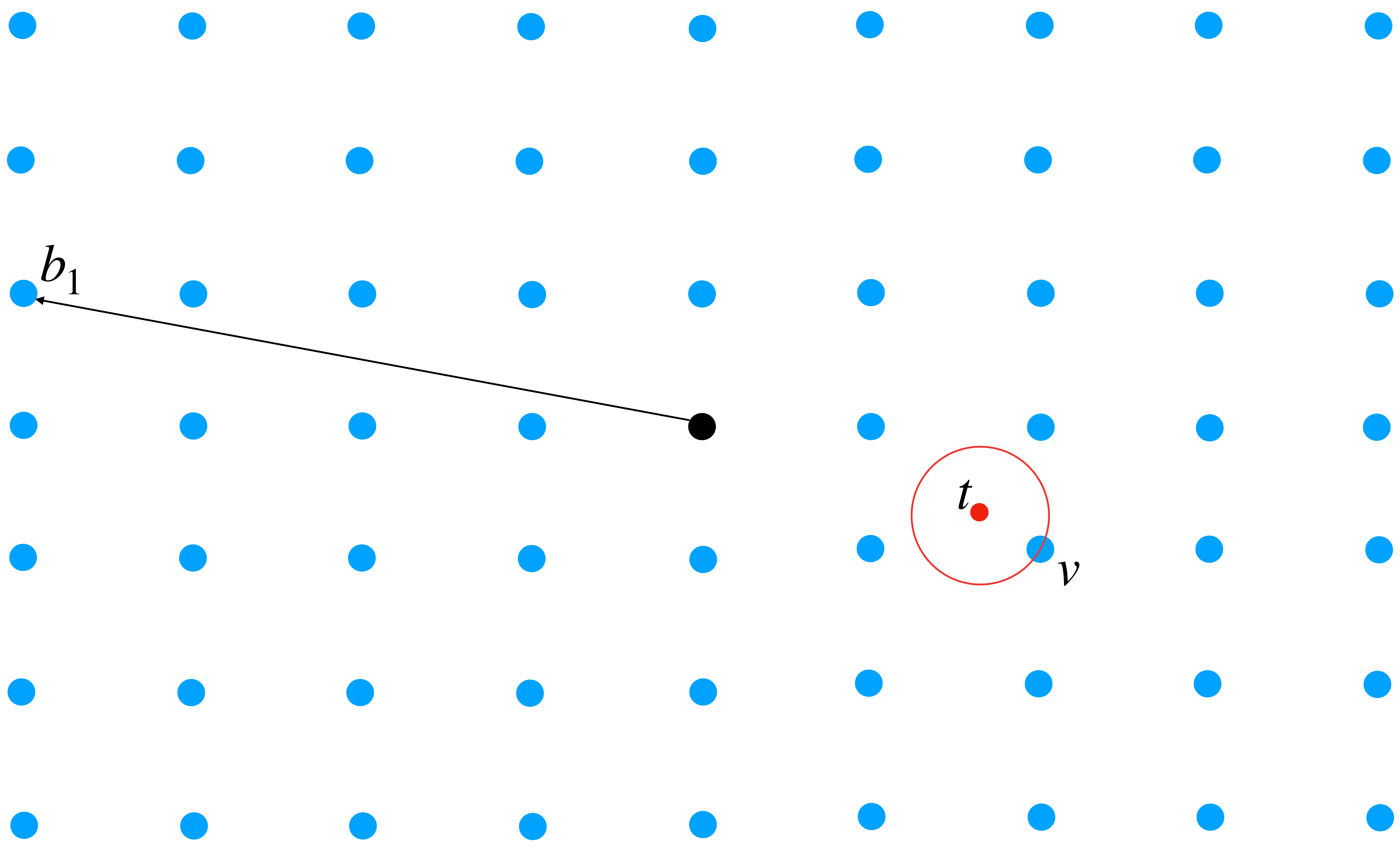



t

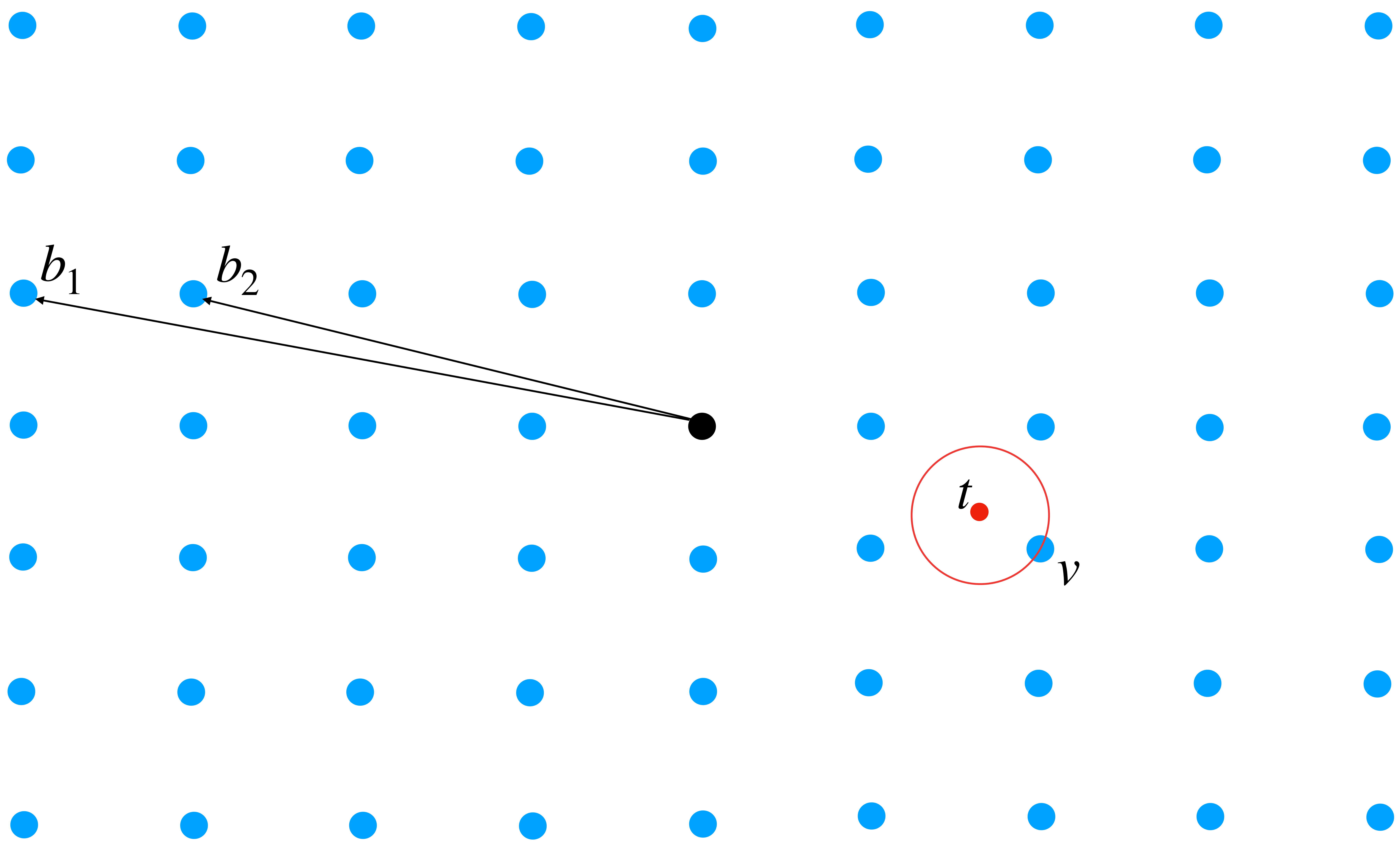

t
v



t
v



b_1
t
v



b_1
b_2
t
v

Facts about CVP

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Algorithm	Time	Space
Enumeration	$n^{O(n)}$	$poly(n)$
Sieving	$2^{O(n)}$	$2^{O(n)}$
Voronoi	$\tilde{O}(2^{2n})$	$\tilde{O}(2^n)$
Gaussian	$2^{n+o(n)}$	$2^{n+o(n)}$

Other lattice problems

Other lattice problems

- Given a basis B, the **Shortest Vector Problem (SVP)** asks for a shortest non-zero vector $v \in \mathscr{L}(B)$, i.e., $||v|| \leq ||u||$ for all $u \in \mathscr{L}(B) \backslash \{0\}$.

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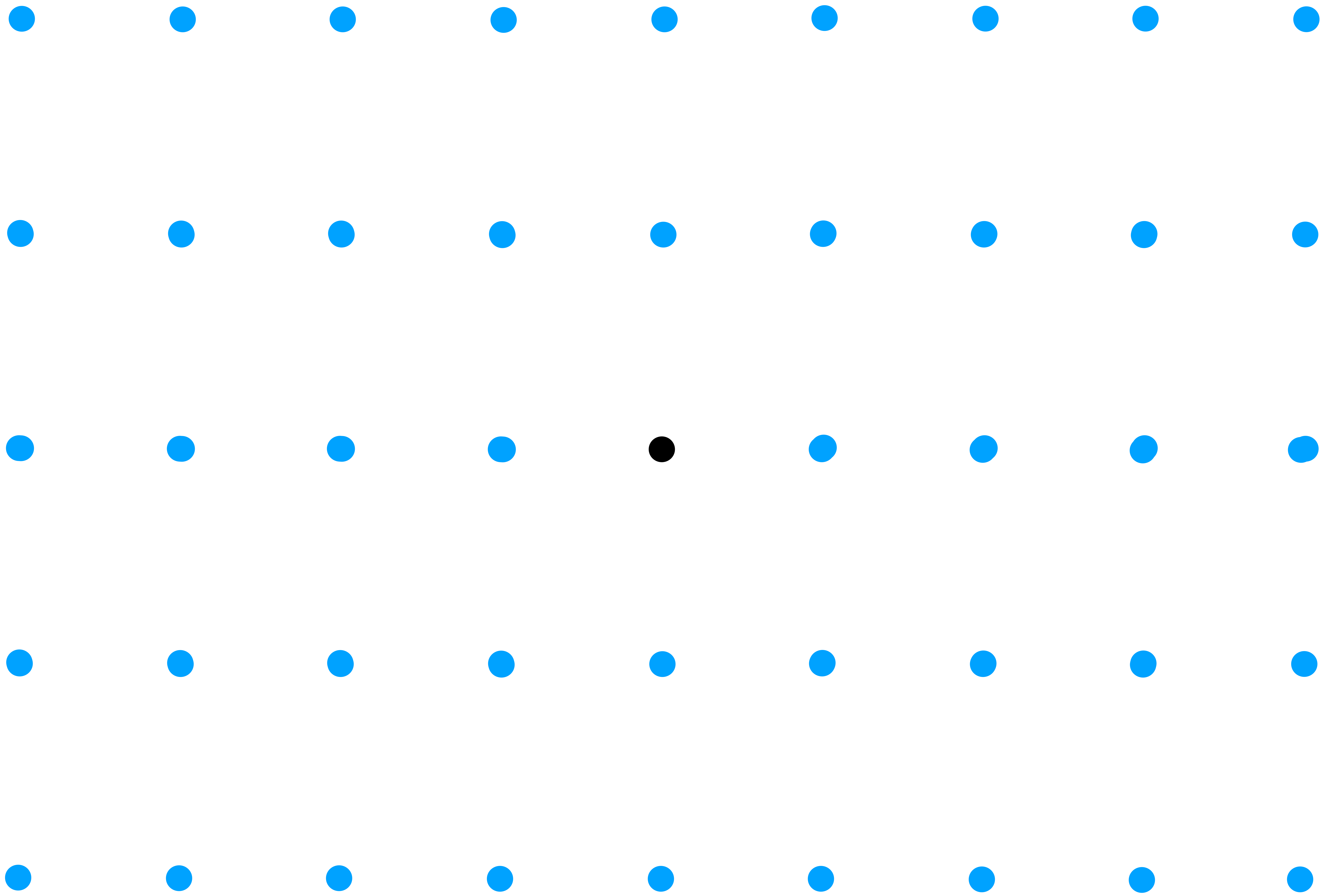
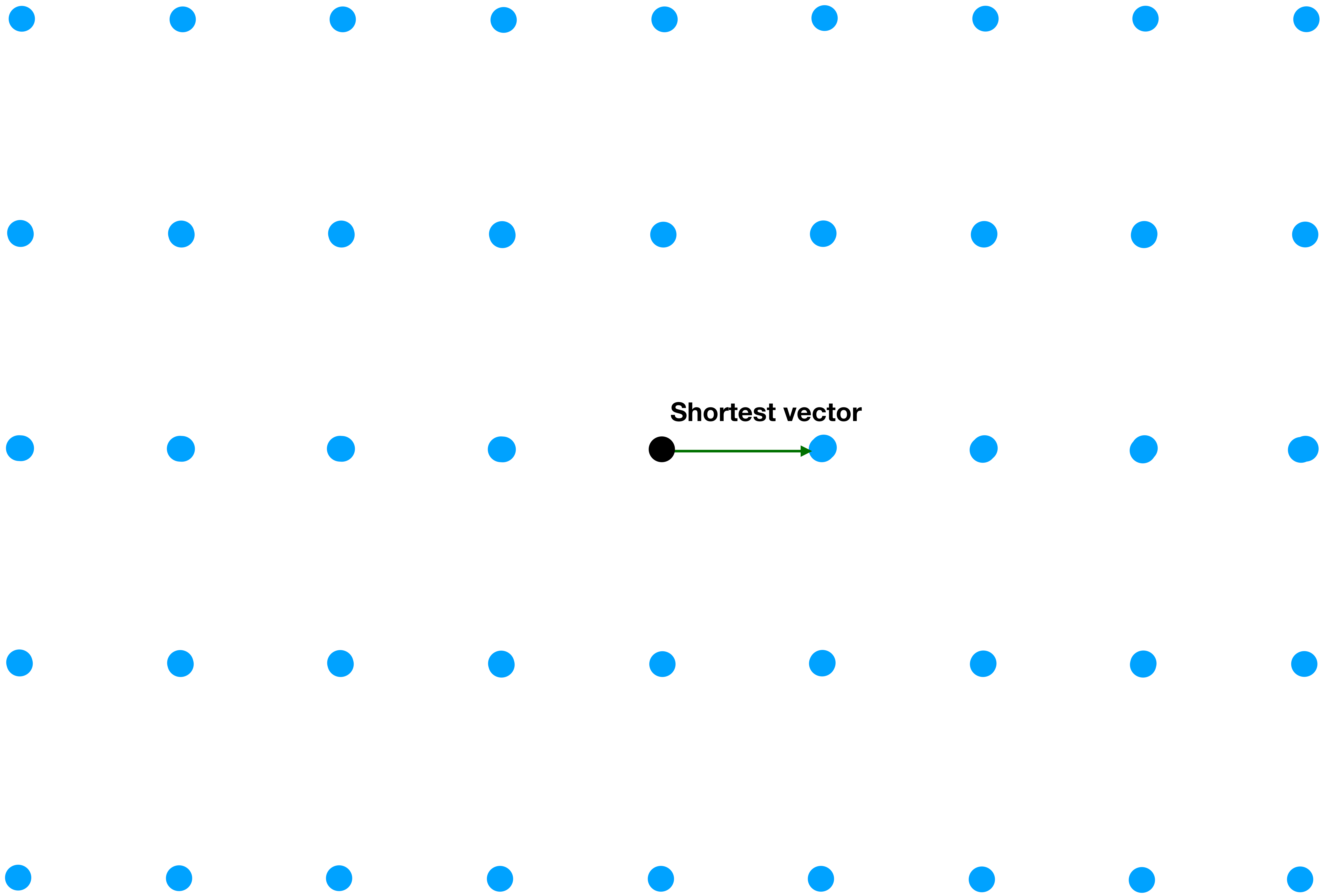
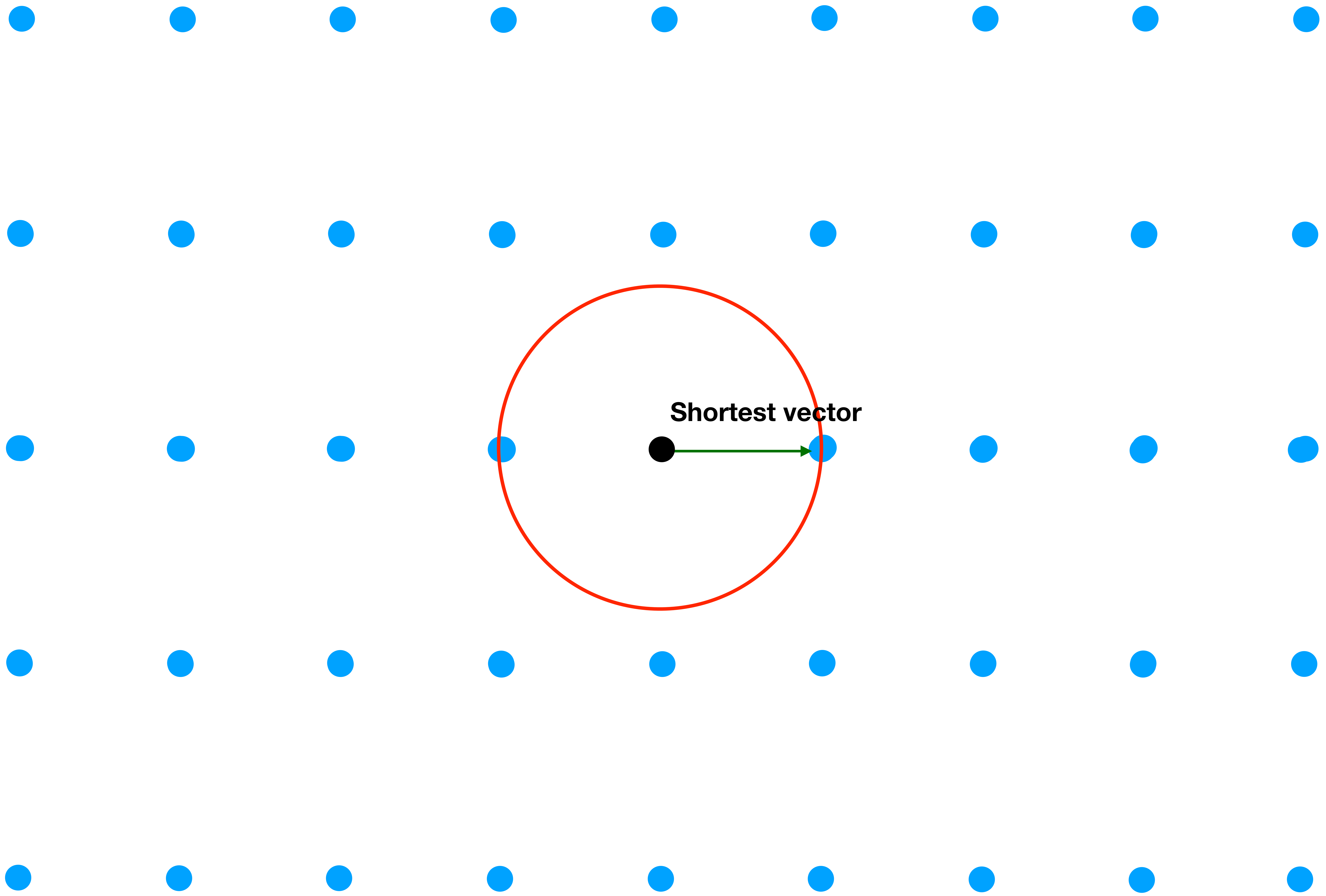
where $\mathscr{B}(x, y)$ is the sphere entered at x with radius y.




Shortest vector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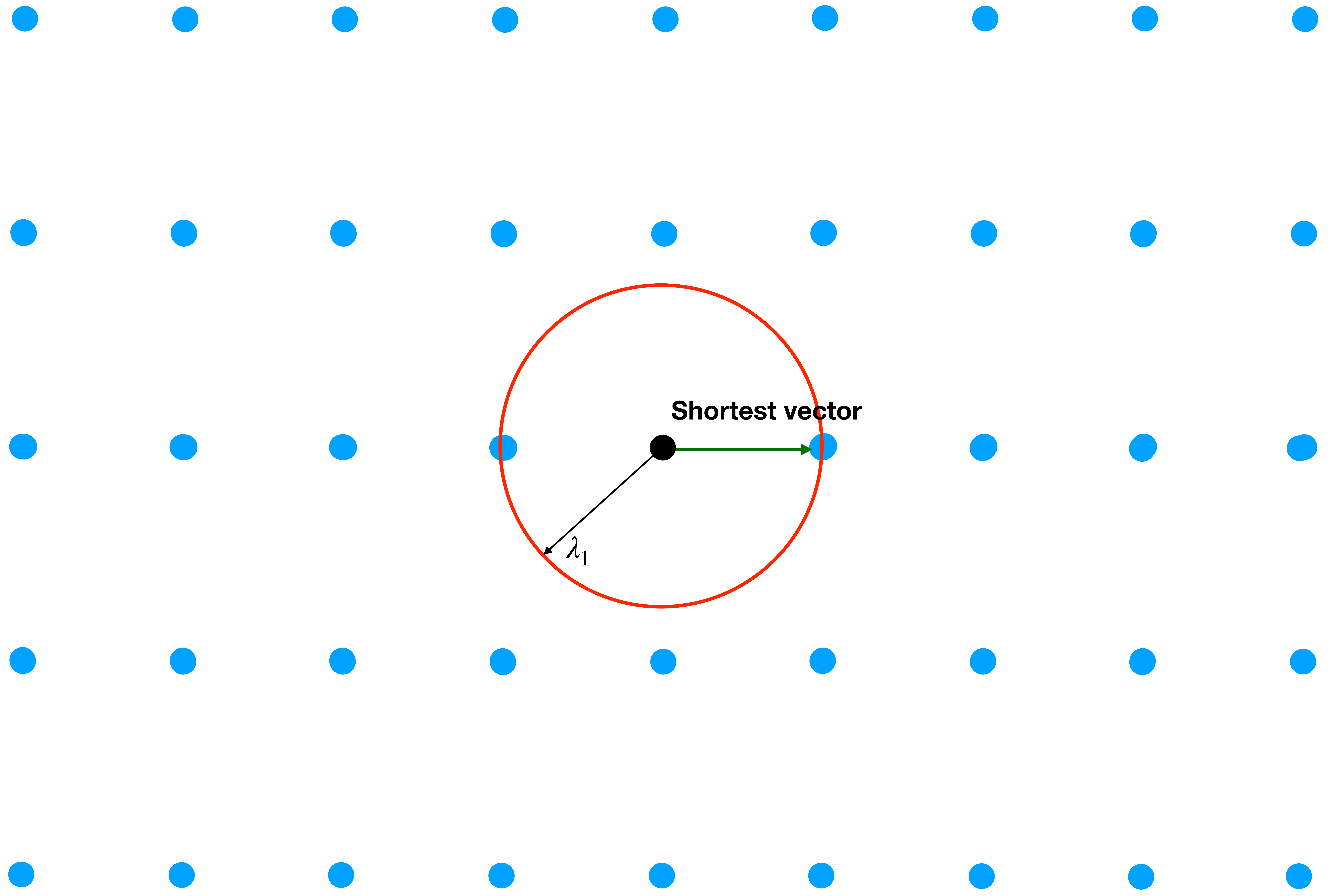
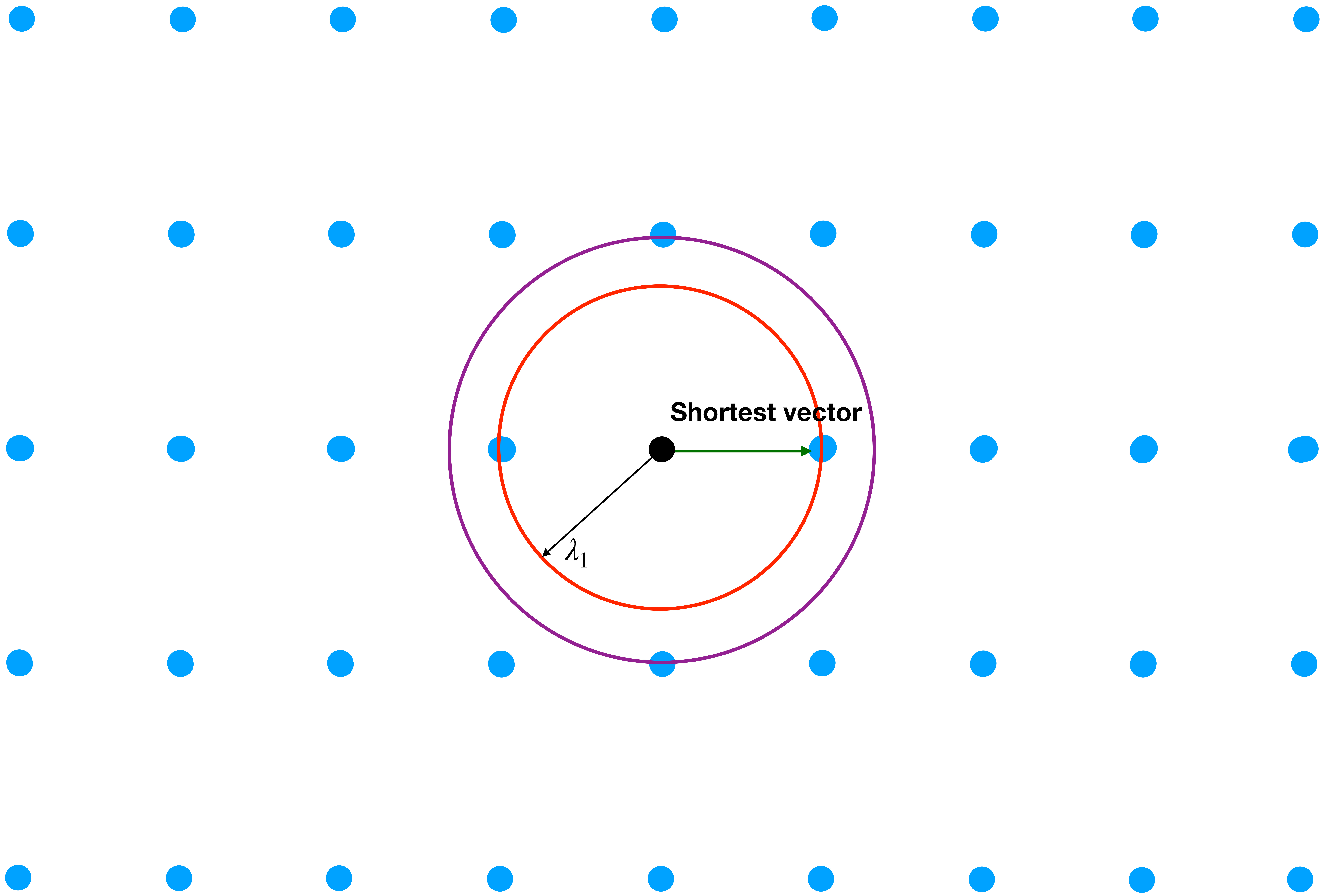

Shortest vector
λ_1


Shortest vector
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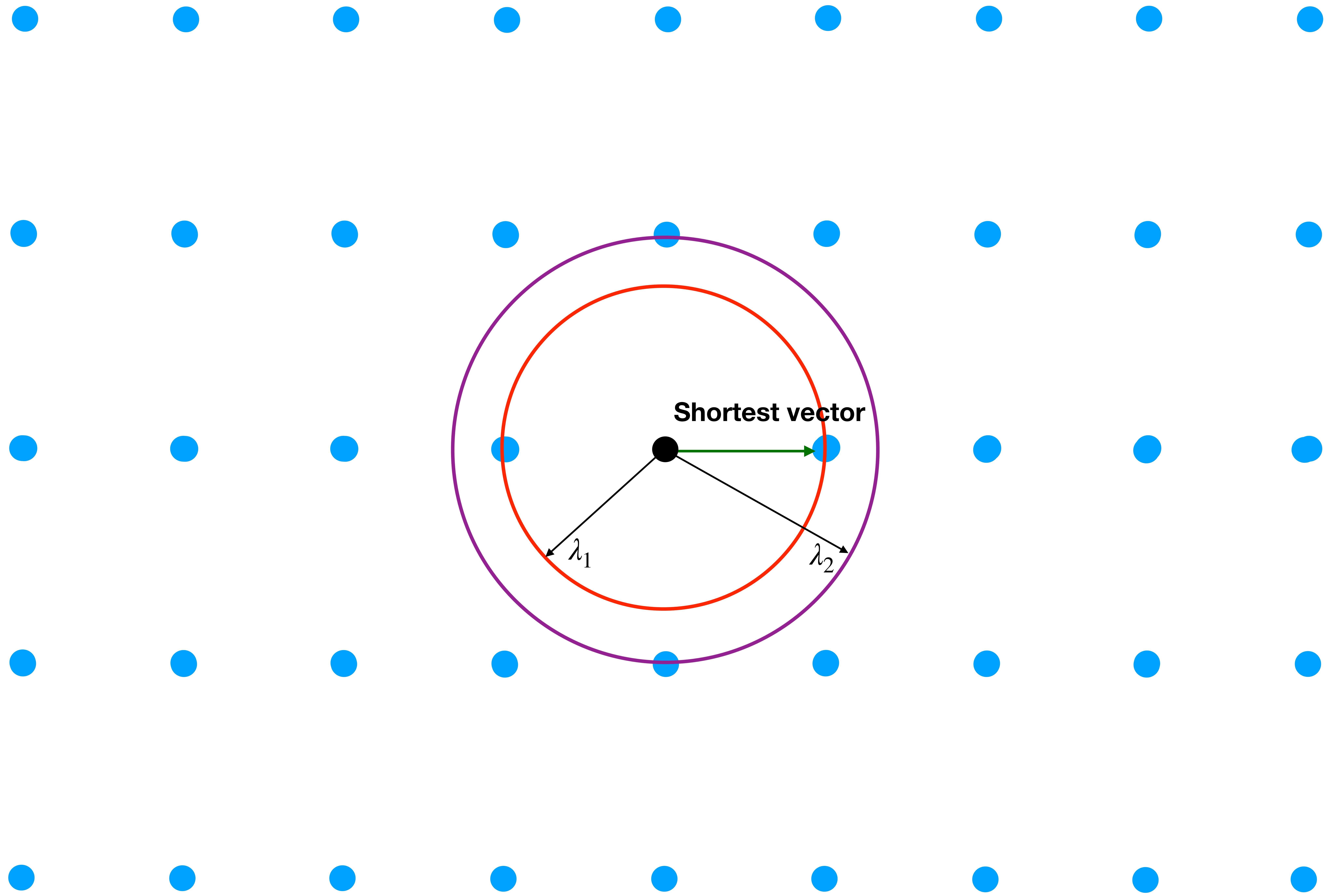
Shortest vector
λ1
λ2

Voronoi cell of a lattice

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The Voronoi cell of a lattice $\mathscr{L}$ is defined as

$$\mathscr{V}(\mathscr{L}) = \{x \in \mathbb{R}^n \mid \forall v \in \mathscr{L} \backslash \{0\}, ||x|| \leq ||x - v||\}$$

In other words, it is set of all points that are closer to the origin than all other non-zero lattice vectors.

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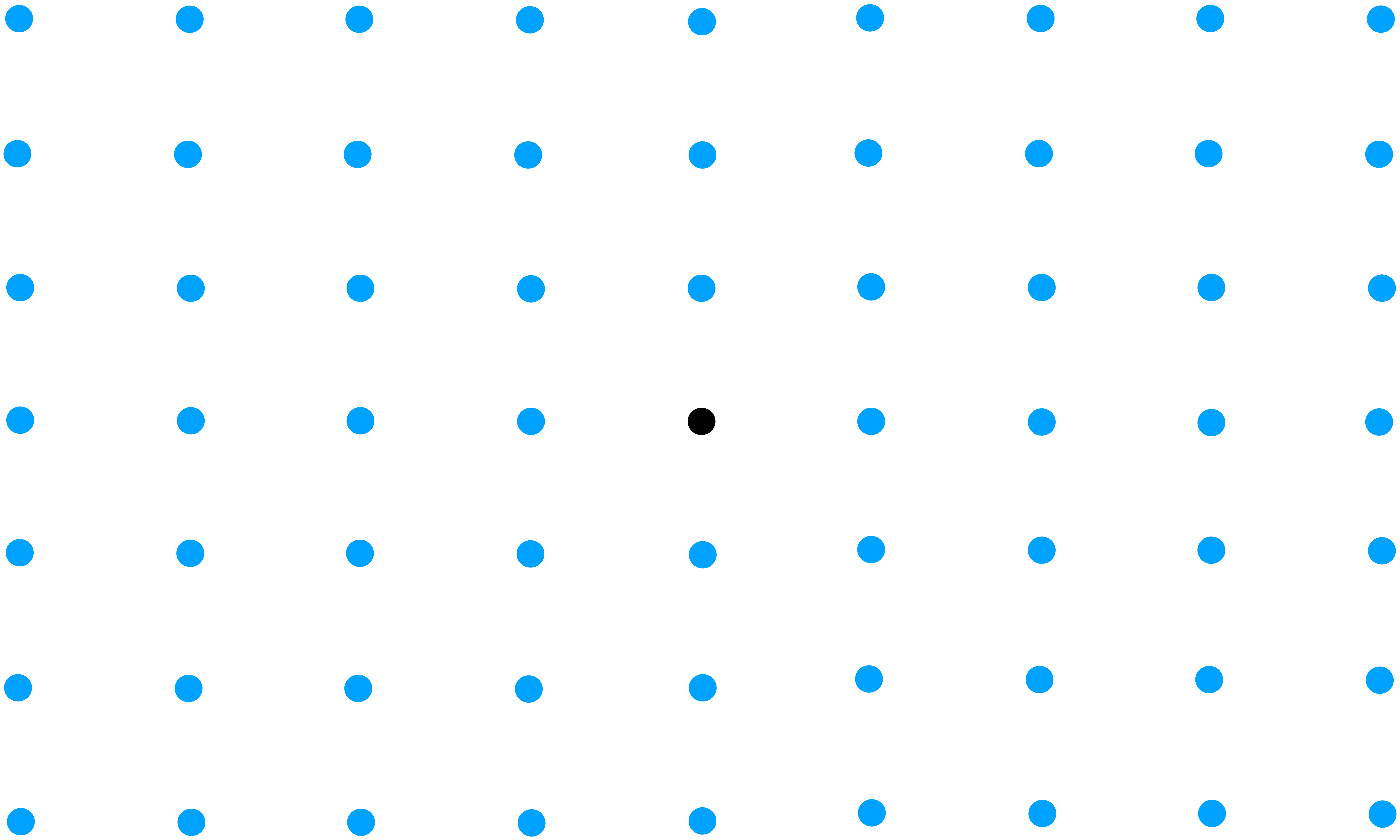
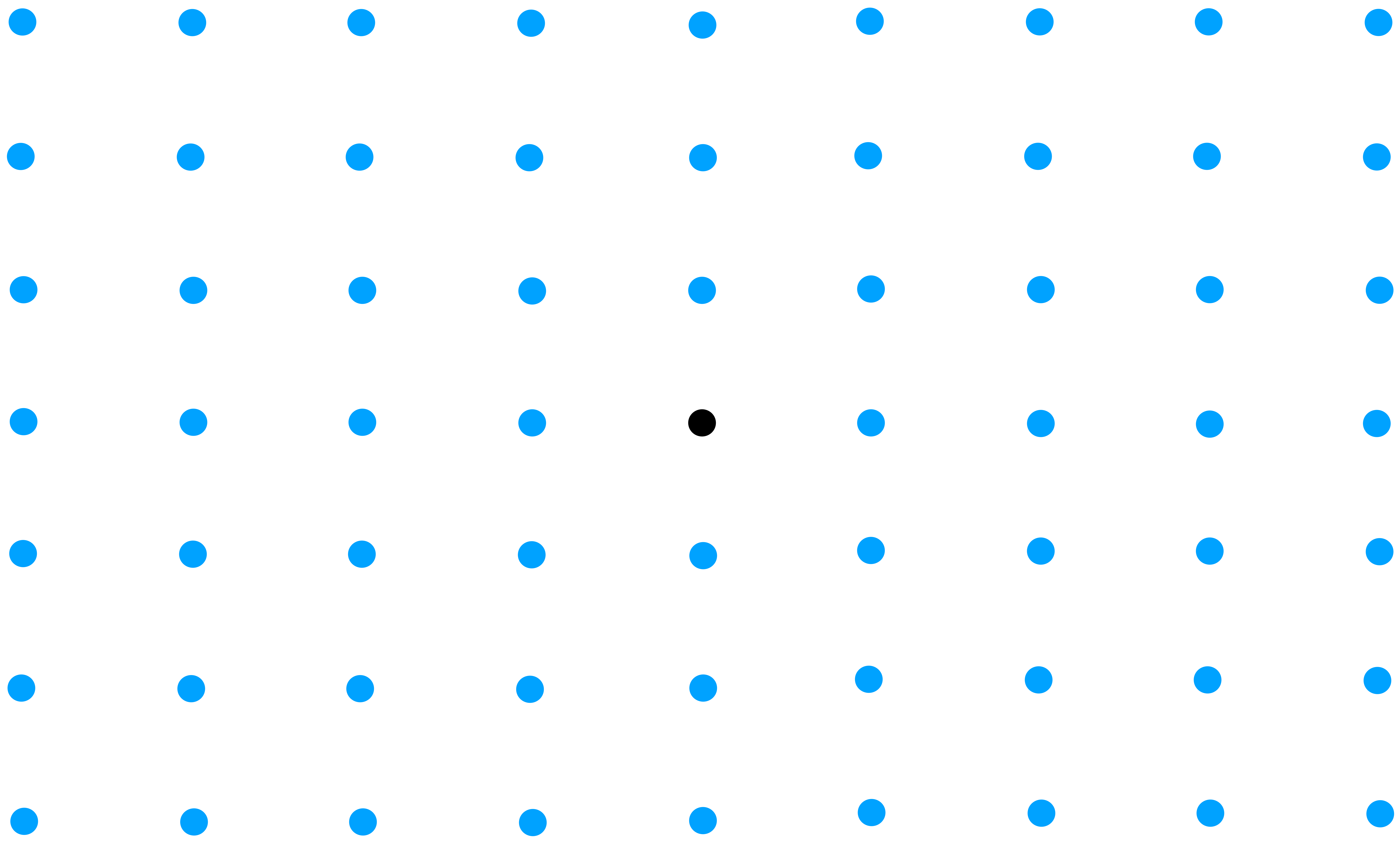
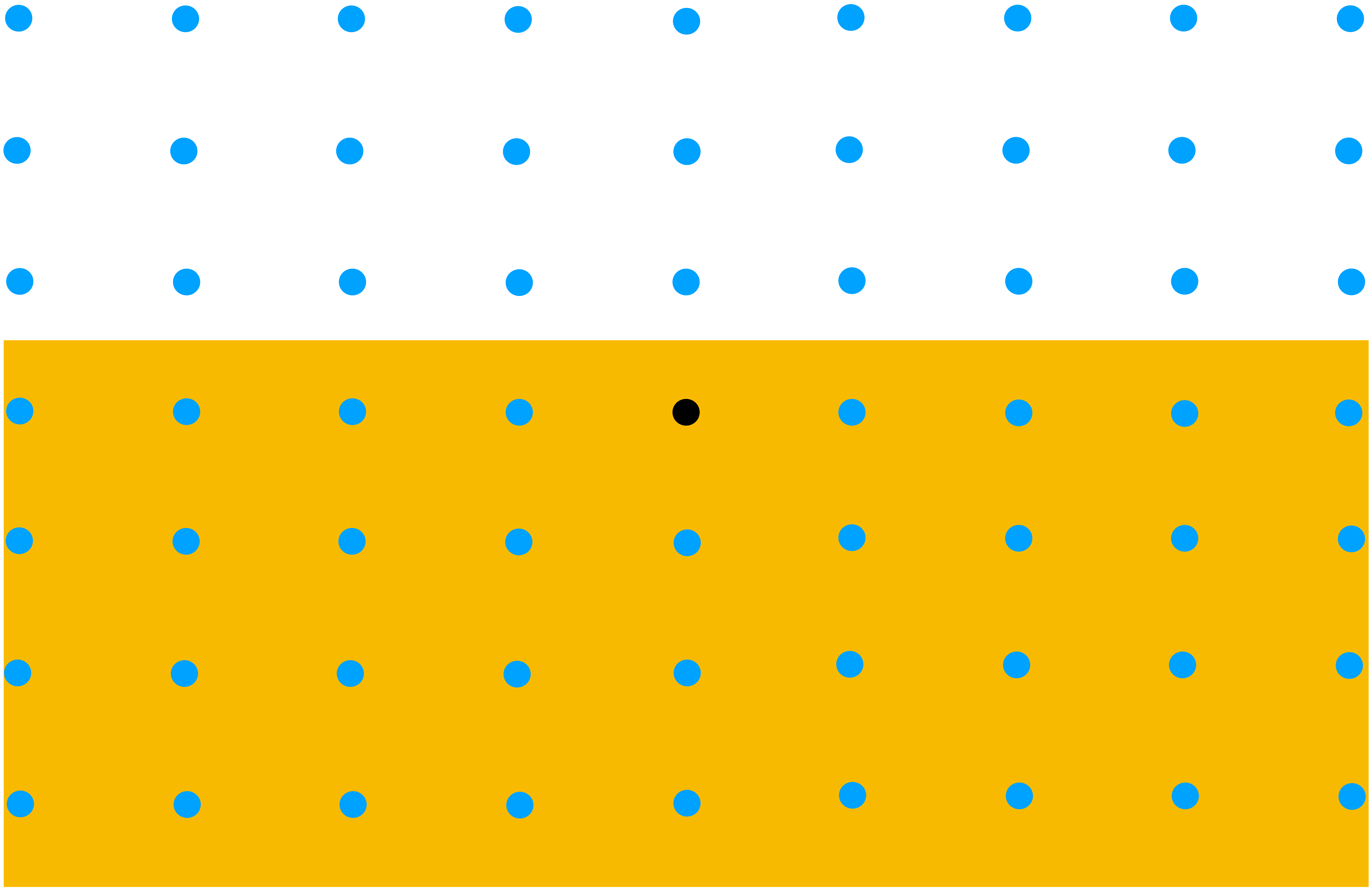
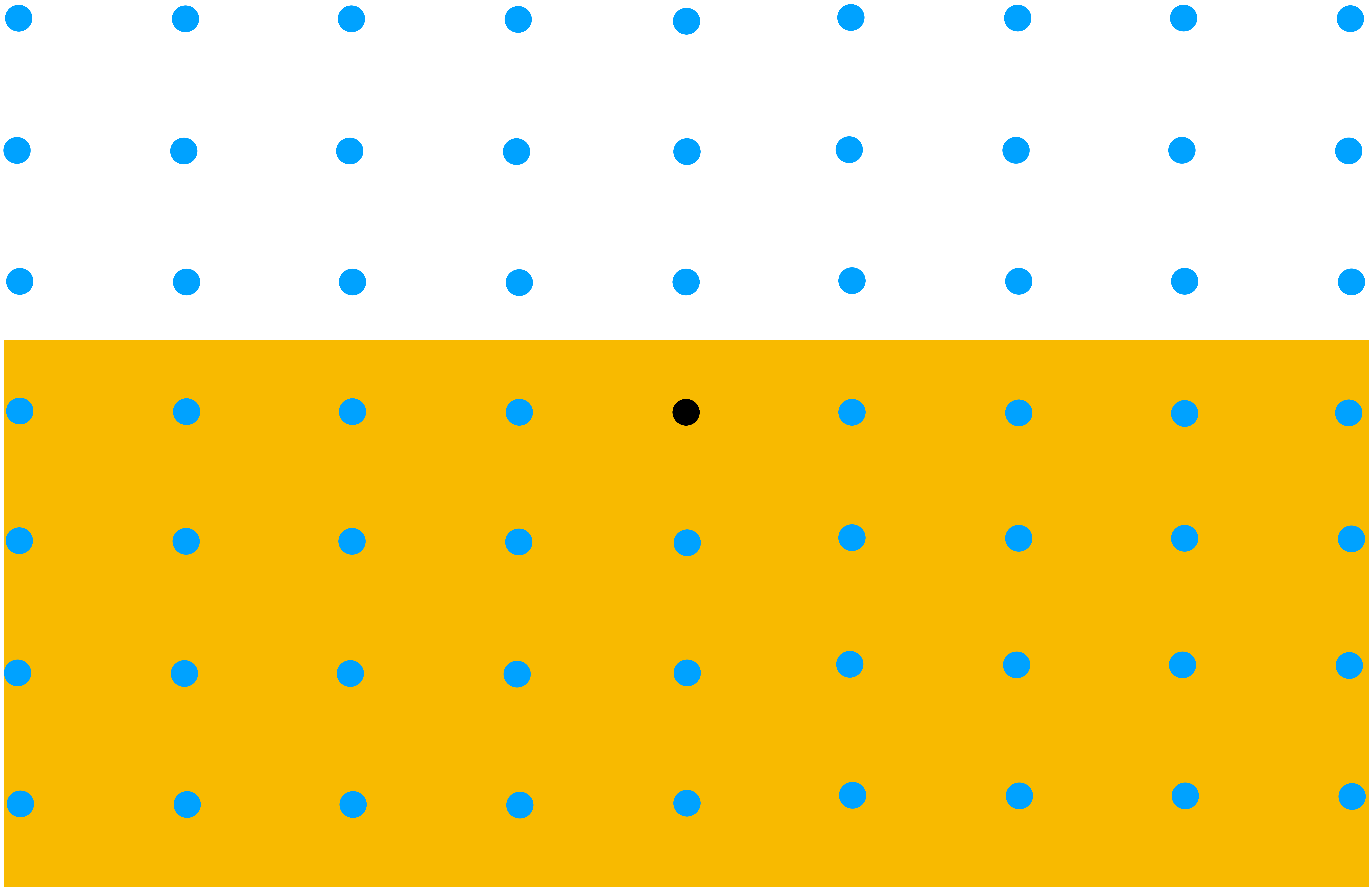
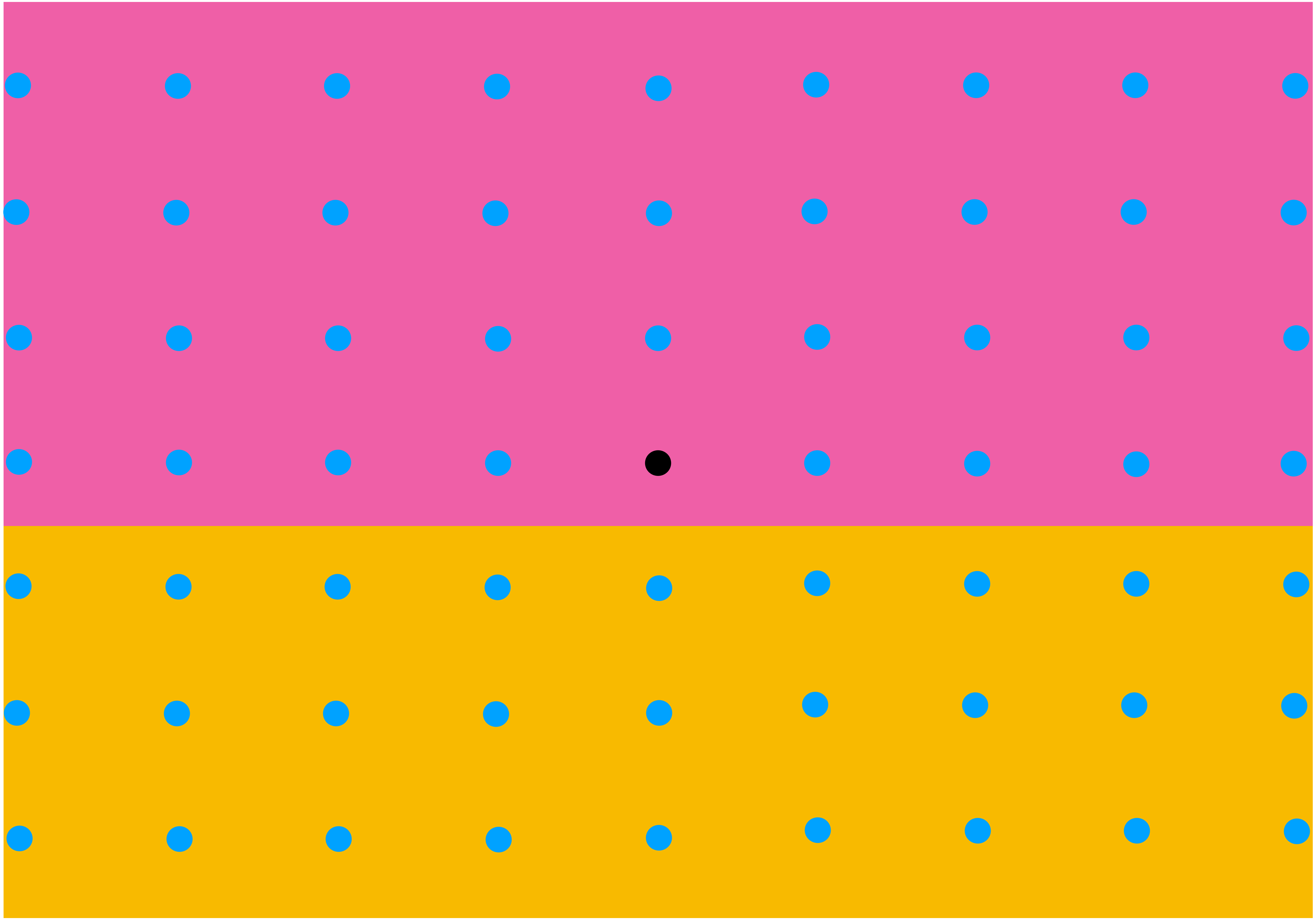
There is a minimal set of lattice vectors called Voronoi relevant vectors $V(L)$ such that $\mathscr{V}(\mathscr{L}) = \bigcap_{v \in V(\mathscr{L})} H(v)$.

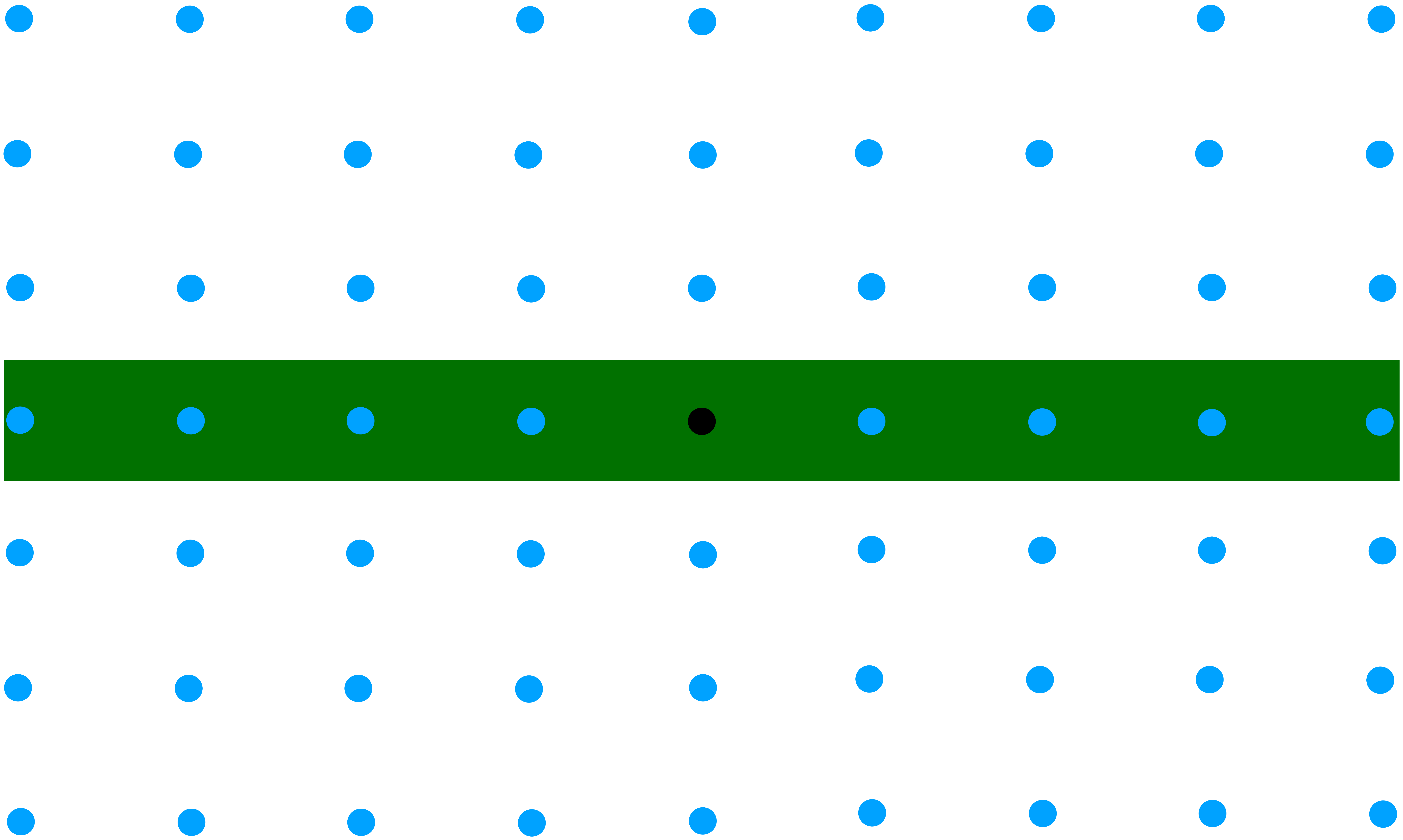


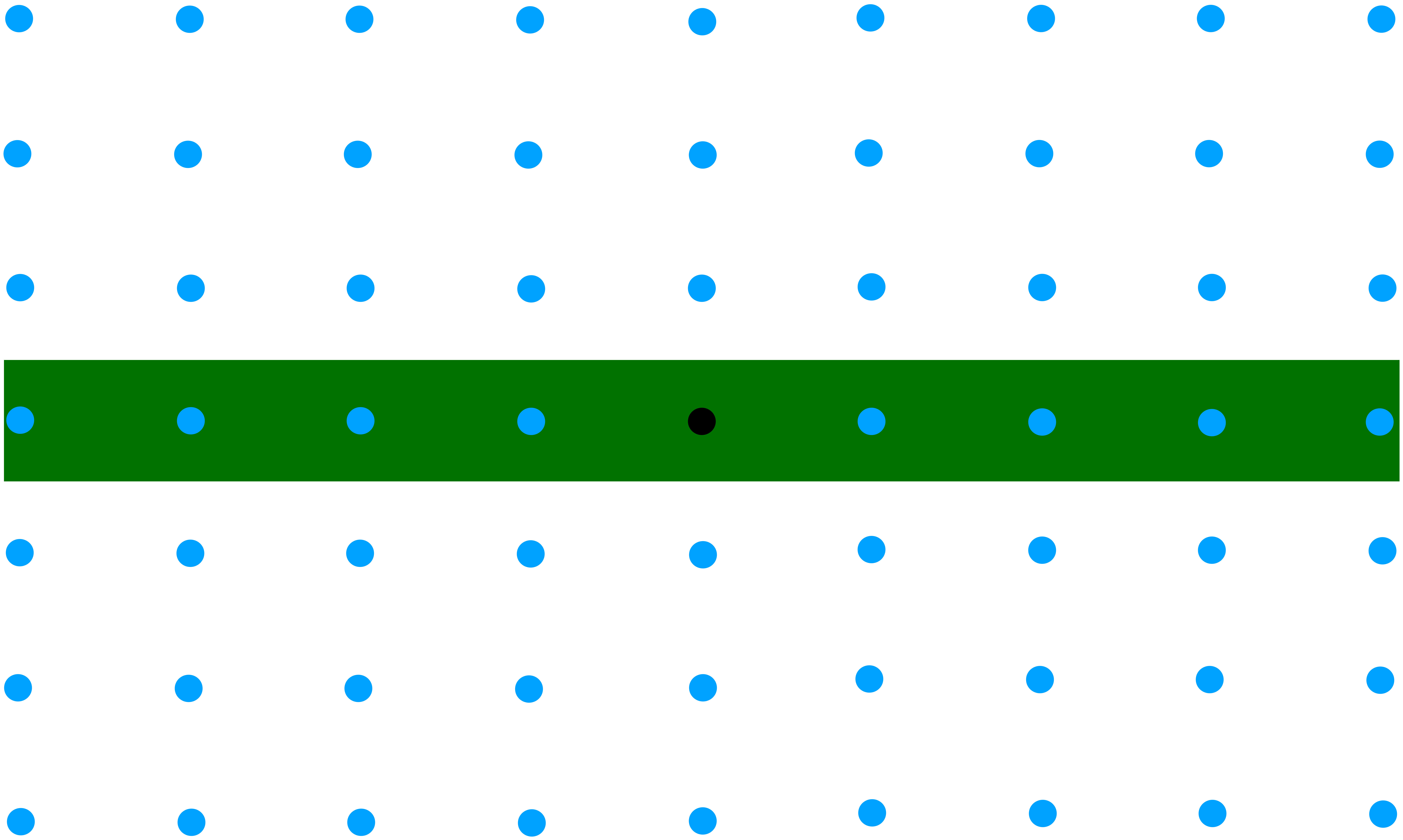


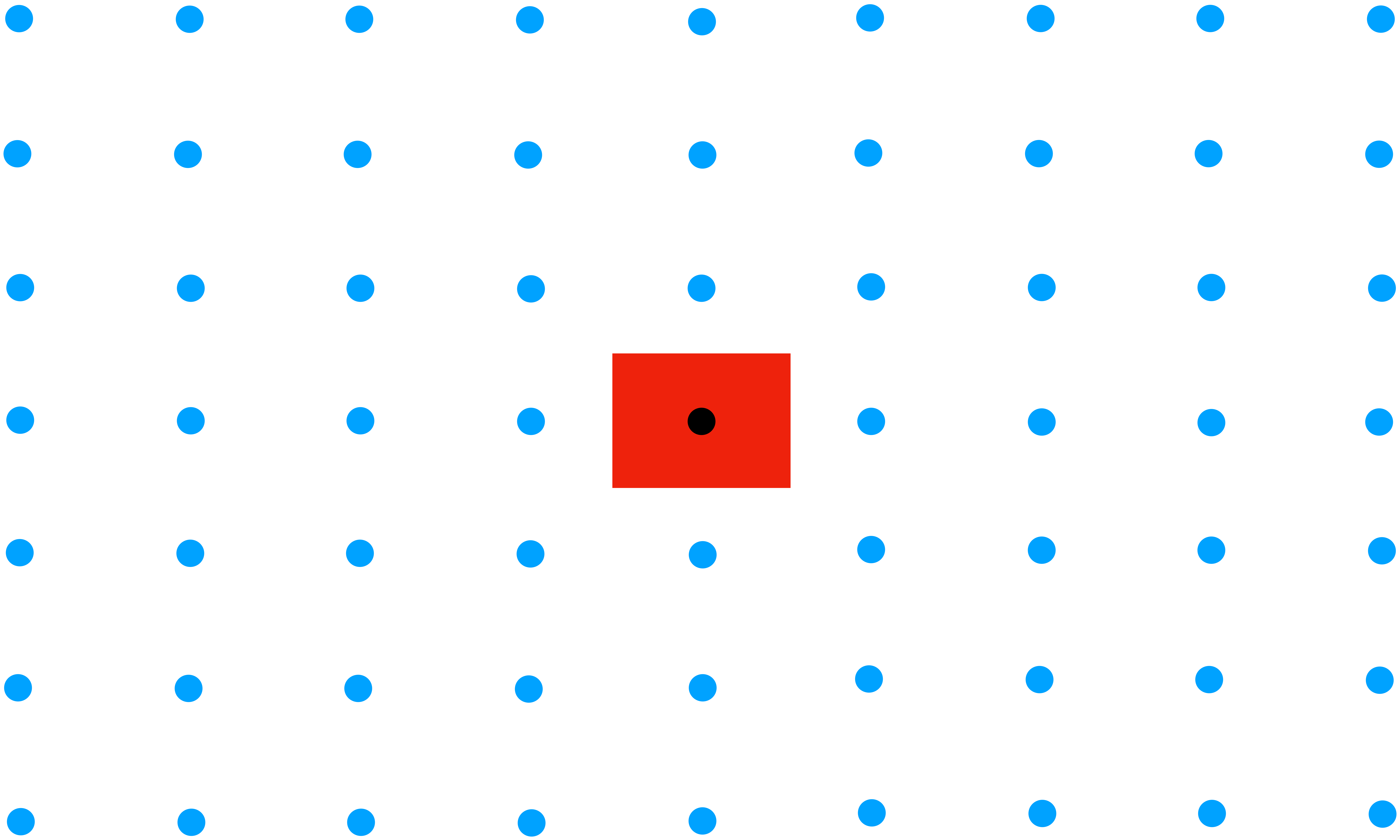


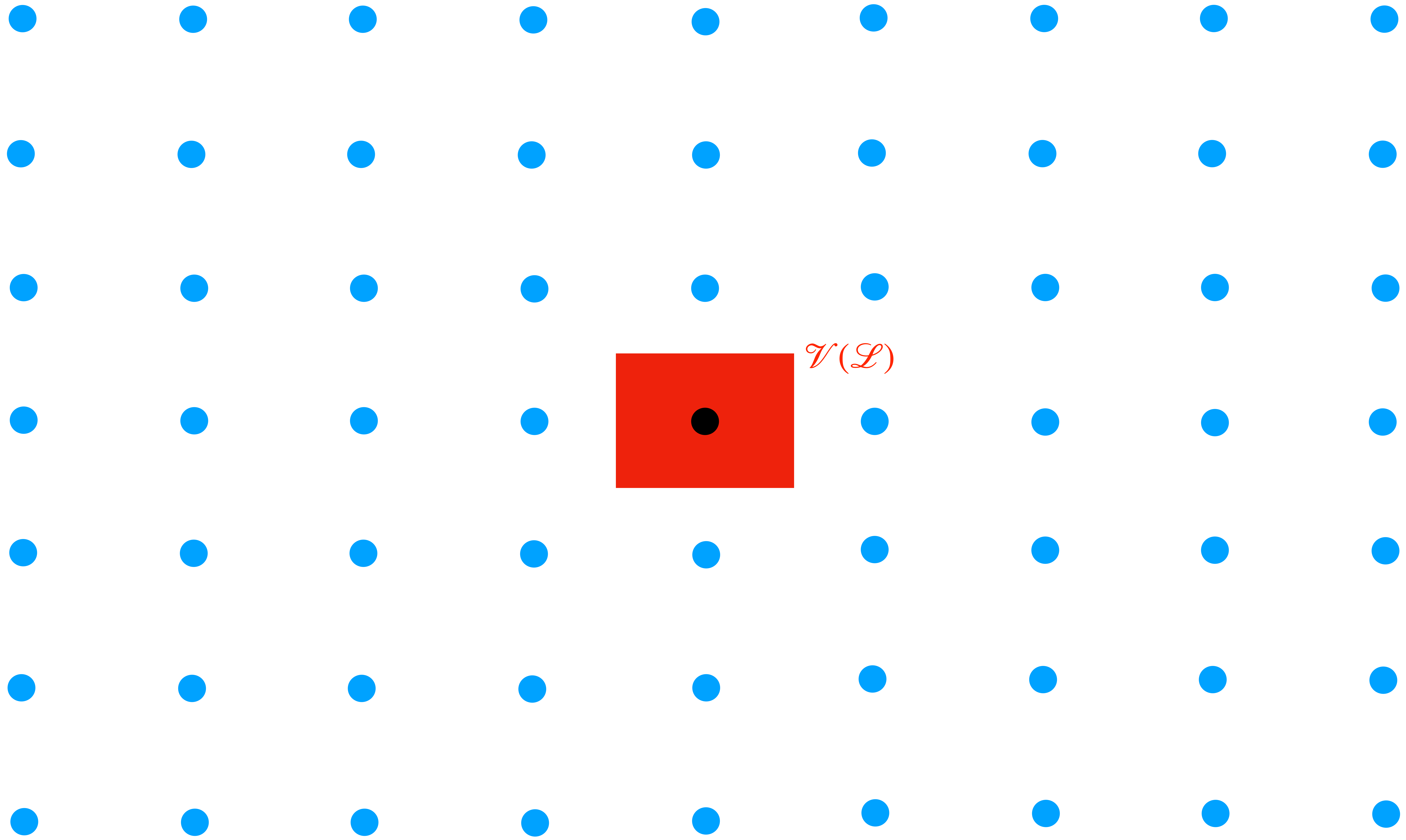


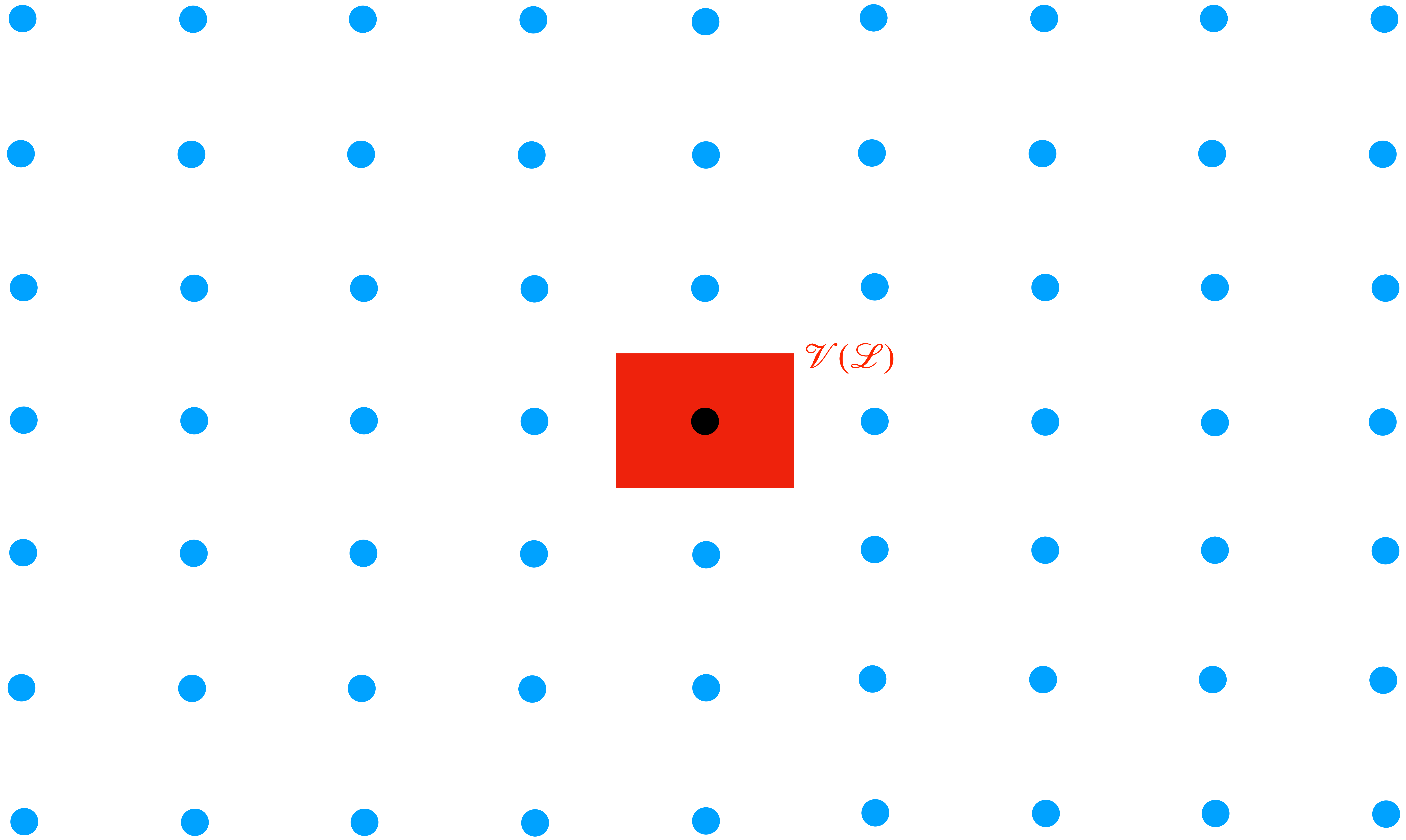









$\mathscr{V}(\mathscr{L})$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Extension Lemma

Preliminaries

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- $\mathbb{Z}^n$ is the lattice spanned by $\{e_1, e_2, \ldots, e_n\}$.
- A vector v in a lattice $\mathscr{L}$ is primitive if $\forall k > 1, v/k \notin \mathscr{L}$.
- B is a basis of $\mathbb{Z}^n \iff B$ is unimodular.
  ($BC = I \implies det(B)det(C) = 1$. But, both
  $B, C \in \mathbb{Z}^{n \times n} \implies det(B) = \pm 1$).

Main Theorem

Main Theorem

Let $v \in \mathbb{Z}^n$ be a primitive vector such that $||v||^2 > 1$. Then, there exists a unimodular matrix $B = \{b_1, b_2, \ldots, b_n\}$ such that $b_n = v$ and $||v||^2 > ||b_i||^2, \forall i \in [n-1]$.

Proof of Main Theorem

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  - $B = [b_1, v]$ is unimodular.
  - We can find c, d such that $|c| < |b|, |d| < |a|$.

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  - Consider $b'_n = [v_1, v_2, \ldots, v_{n-1}]$
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  - $B = \begin{bmatrix} 0 & B' \\ 1 & 0 \end{bmatrix}$

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- Case 3: v has at least one component as 1.

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  - Let $r_i, s_i \in \mathbb{Z}$ such that $r_i v_i + s_i d_{i-1} = d_i$.
  - Let $T_2 = \begin{bmatrix} 1 & 0 & \ldots & 0 & 0 \\ 0 & 1 & \ldots & 0 & 0 \\ \vdots & & & & \\ 0 & 0 & \ldots & r_2 & s_2 \\ 0 & 0 & \ldots & -d_1/d_2 & v_2/d_2 \end{bmatrix}$ where $d_1 = v_1$.

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  - $T_2 v = [v_n, v_{n-1}, \ldots, d_2, 0]^T$

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  - $v = T_2^{-1} \dots, T_{n-1}^{-1} T_n^{-1} e_1$

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- $$B = T_2^{-1}\ldots, T_{n-1}^{-1}T_n^{-1} = \begin{bmatrix} v_n & -s_n & 0 & \ldots & 0 & 0 \\ v_{n-1} & \frac{v_{n-1}r_n}{d_{n-1}} & -s_{n-1} & \ldots & 0 & 0 \\ v_{n-2} & \frac{v_{n-2}r_n}{d_{n-1}} & \frac{v_{n-2}r_{n-1}}{d_{n-2}} & \ldots & 0 & 0 \\ \vdots & & & & & \\ v_2 & \frac{v_2r_n}{d_{n-1}} & \frac{v_2r_{n-1}}{d_{n-2}} & \ldots & \frac{v_2r_3}{d_2} & -s_2 \\ v_1 & \frac{v_1r_n}{d_{n-1}} & \frac{v_1r_{n-1}}{d_{n-2}} & \ldots & \frac{v_1r_3}{d_2} & r_2 \end{bmatrix}$$

Successive Minima from Voronoi Relevant Vectors

Main theorem

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Given a basis $B = \{b_1, \ldots, b_n\}$, the **Successive Minima Problem (SMP)** ask for n linearly independent vectors $\{s_1, \ldots, s_n\} \subseteq \mathscr{L}(B)$ such that $||s_i|| = \lambda_i(\mathscr{L}(B))$.

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2) We can modify the algorithm given by Micciancio and Voulgaris to find a solution to SMP without using CVP oracles.

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- We showed how to construct a basis for $\mathbb{Z}^n$ from a primitive vector v such that the rest of the basis vectors are strictly shorter than v.
- Discussed that a solution to SMP is contained in the set of Voronoi relevant vectors.
- Is it possible to extend $v_1, v_2, \ldots, v_k$ to a basis $[v_1, \ldots, v_k, b_{k+1}, \ldots, b_n]$ of $\mathbb{Z}^n$ such that every b_i's are strictly shorter than the longest v_j.

Thank You !

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- Let us first show that all shortest vector belongs to $V(\mathscr{L})$. Assume the contrary.
  - $||s/2 - w|| < ||s/2|| \implies ||s - 2w|| < ||s||$
  - $||s/2 - w|| = ||s/2|| \implies cos(\theta) = ||w||/||s||$. Since, $||w|| \geq ||s|| \implies cos(\theta) \geq 1$. But, this implies $w = s$.

- Assume that $s_1, \ldots, s_{i-1} \in V(\mathscr{L})$ and $s_i \notin V(\mathscr{L})$ for some i.

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  - $||s_i - 2w|| < ||s_i||$: We can show that $s_i - 2w \in Span(s_1, \ldots, s_{i-1})$.
    $||w|| = ||w - s_i/2 + s_i/2|| < ||s_i|| \implies w \in Span(s_1, \ldots, s_{i-1})$.
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  - $||s_i - 2w|| = ||s_i||$:
    $||w||^2 = < s_i, w > \implies cos(\theta) = ||w||/||s_i||$. $\theta \neq 0$, therefore
    $||s_i|| > ||w||$ and $w \in Span(s_1, \ldots, s_{i-1})$. Also,
    $||s_i - w||^2 = ||s_i||^2 - ||w||^2 < ||s_i||^2$. Therefore,
    $s_i - w \in Span(s_1, \ldots, s_{i-1}) \implies s_i \in Span(s_1, \ldots, s_{i-1})$.

Thanks again!